SELMER GROUPS AND CENTRAL VALUES OF L-FUNCTIONS FOR MODULAR FORMS

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Abstract. In this article, we construct an Euler system using CM cycles on Kuga-Sato varieties over Shimura curves and show a relation with the central values of Rankin-Selberg L-functions for elliptic modular forms and ring class characters of an imaginary quadratic field. As an application, we prove that the non-vanishing of the central values of Rankin-Selberg L-functions implies the finiteness of Selmer groups associated to the corresponding Galois representation of modular forms under some assumptions.

Introduction

Let ℓ be a prime and fix an embedding $\iota_\ell : \overline{\mathbb{Q}} \to \mathbb{C}_\ell$, where $\mathbb{C}_\ell = \widehat{\overline{\mathbb{Q}_\ell}}$. Let N be a positive integer and k an even positive integer. Let

$$f = \sum_{n=1}^{\infty} a_n(f) e^{2\pi i n z} \in S_k(\Gamma_0(N))^{\mathrm{new}}$$

be a normalized cuspidal eigenform. Denote $E = \mathbb{Q}_\ell(\{a_n(f)\}_n)$ for the Hecke field of f over $\mathbb{Q}_\ell$ and fix a uniformizer λ of the ring of integers $\mathcal{O}$ of E. Denote the residue field of E by $\mathbb{F}$. Let

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(E)$$

be the Galois representation associated to f. We put $\rho_f^* = \rho_f \otimes E(\frac{2-k}{2})$ and denote V_f for the representation space of ρ_f^*. Fix a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $\mathcal{O}$-lattice T_f and set $A_f = V_f/T_f$. Let L be an abelian extension of $\mathbb{Q}$ and χ a character of the Galois group $\mathrm{Gal}(L/\mathbb{Q})$. By the Bloch-Kato conjecture [5], it is expected that the central value of the L-function of f twisted by the character χ is related to the order of the χ-part of the Selmer group $\mathrm{Sel}(L, A_f)$. Kato [21] proved that the non-vanishing of the central value $L(f, \chi, k/2)$ implies the finiteness of the χ-part of the Selmer group $\mathrm{Sel}(L, A_f)$. Moreover, Kato showed a result on the upper bound of the size of Selmer group in terms of the special values of L-functions using the Euler system of Beilinson-Kato elements in K_2 of modular curves. For an elliptic curve E over $\mathbb{Q}$ and an imaginary quadratic filed K, similar results in the anticyclotomic setting are considered by Bertolini-Darmon [3] and Longo-Vigni [23] using the Euler system constructed from CM points on Shimura curves. These results were generalized to modular abelian varieties over totally real fields by Longo [22] and Nekovář [27]. In this paper, we will consider the generalization of these results for the central values of L-function associated to higher weight modular forms twisted by ring class characters over an imaginary quadratic field K,

We fix an imaginary quadratic field K of discriminant $D_K < 0$ satisfying $(N, D_K) = 1$ and denote the integer ring of K by $\mathcal{O}_K$. Then K determines a factorization $N = N^+N^-$, where N^+ is divisible only by primes which splits in K and N^- is divisible only by primes which are inert in K. Assume that

(ST) $\qquad N^-$ is a square-free product of an odd number of inert primes.

Fix an integer m such that $(ND_K, m) = 1$ and let K_m be the ring class field of conductor m. Let χ be a character of the Galois group $\mathcal{G}_m = \mathrm{Gal}(K_m/K)$. Then we can define the Rankin-Selberg L-function $L(f/K, \chi, s)$ associated to f and χ. We define a complex number Ω_{f,N^-} by

$$\Omega_{f,N^-} = \frac{4^{k-1}\pi^k ||f||_{\Gamma_0(N)}}{\xi_f(N^+, N^-)},$$

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where $||f||_{\Gamma_0(N)}$ is the Petersson norm of f and $\eta_f(N)$ is the congruence number of f among cusp forms in $S_k(\Gamma_0(N))^{N^- -\mathrm{new}}$(see §1 for details). Then Bloch-Kato conjecture predicts a relation between the values $\frac{L(f/K,\chi,k/2)}{\Omega_{f,N^-}}$ and the size of the χ-part of Selmer group $\mathrm{Sel}(K_m, A_f)$,

We consider the following condition.

Hypothesis (CR$^+$) . (1) $\ell > k+1$ and $\#(\mathbb{F}_\ell^\times)^{k-1} > 5$,

(2) The restriction of the residual Galois representation $\bar{\rho}_f$ of ρ_f to the absolute Galois group of $\mathbb{Q}(\sqrt{(-1)^{\frac{\ell-1}{2}}\ell})$ is absolutely irreducible,

(3) $\bar{\rho}_f$ is ramified at q if either (i) $q \mid N^-$ and $q^2 \equiv 1 \pmod{\ell}$ or (ii) $q || N^+$ and $q \equiv 1 \pmod{\ell}$,

(4) $\bar{\rho}_f$ restricted to the inertia group of $\mathbb{Q}_q$ is irreducible if $q^2 \mid N$ and $q \equiv -1 \pmod{\ell}$.

Our main result is the following theorem.

Theorem 0.1. *Let χ be a ring class character of conductor m. Suppose that f is a cuspidal Hecke eigen newform. Assume the following conditions:*

(1) *ℓ does not divide $ND_K[K_m : K]$.*

(2) *the residual Galois representation $\overline{\rho}_f$ satisfies the condition* (CR$^+$).

If $c = \mathrm{ord}_\lambda(\frac{L(f/K,\chi,k/2)}{\Omega_{f,N^-}})$ is finite, then we have $\lambda^{\frac{c}{2}} \cdot \mathrm{Sel}(K_m, A_f)^\chi = 0$. In particular, if $L(f/K, \chi, k/2)$ is not zero, then for all but finitely many primes λ the χ-part of the Selmer group $\mathrm{Sel}(K_m, A_f)$ *is trivial.*

Remark 0.2. (1) The assumption (ST) implies that f is not a CM form. Hence the residual Galois representation $\overline{\rho}_f = \overline{\rho}_{f,\lambda}$ satisfies the condition (CR$^+$) for all but finitely many λ.

(2) Let Ω_f^{can} be Hida's canonical period defined by

$$\Omega_f^{\mathrm{can}} = \frac{4^{k-1}\pi^k ||f||_{\Gamma_0(N)}}{\eta_f(N)},$$

where $\eta_f(N)$ is the congruence number of f among cusp forms in $S_k(\Gamma_0(N))$. Under the hypothesis (CR$^+$), one can show that

$$\Omega_{f,N^-} = u \cdot \Omega_f^{\mathrm{can}} \quad \text{for some } u \in \mathcal{O},$$

if we further assume that $\overline{\rho}_f$ is ramified at all primes dividing N^-.

A similar result is given as a corollary of anticyclotomic Iwasawa main conjecture concerned in [9] under the ordinary condition. In this paper, we remove the ordinary condition.

To prove our main theorem, we develop the method of Bertolini-Darmon [3] on the Euler system obtained from CM points on Shimura curves. In [9], we used an Euler system obtained from CM points on Shimura curves and congruences between modular forms of higher weight and modular forms of weight two in the ordinary case. However, in the non-ordinary case it seems difficult to use such congruences. Therefore we choose to use CM cycles on Kuga-Sato varieties over Shimura curves instead of CM points. For the construction of Euler system, we also use a level raising result (Theorem 5.3) for higher weight modular forms and the assumption (CR$^+$) is necessary to show the level raising result. More precisely, under the assumption (CR$^+$) we have a freeness result (Proposition 5.1) of the space of definite quaternionic modular forms as Hecke modules and it is used in an important step in the proof of the level raising result. The freeness result is a generalization of [8, Proposition 6.8] to the "low weight crystalline case" which is closely related to "$R = \mathbb{T}$" theorems and our case was considered by Taylor [33]. Then one can construct an Euler system using CM cycles and a level raising argument.

Moreover we show a relation between the Euler system and central values of Rankin-Selberg L-functions (Theorem 7.4), so-called the first explicit reciprocity law by Bertolini-Darmon. In the case of weight 2, the explicit reciprocity law is proved by Kummer theory and the theory of ℓ-adic uniformization of Shimura curves. To show the explicit reciprocity law in higher weight case, it is necessary to compute the image of CM cycles under the ℓ-adic Abel-Jacobi map which is defined by Hochschild-Serre spectral sequence. Since it is difficult to compute the image of CM cycles directly, we give a different description of the image of CM cycles using the theory of vanishing cycles and the theory of ℓ-adic uniformization of Shimura curves. This is a main ingredient of our proof.

This article is organized as follows. First, we review the theory of modular forms on quaternion algebras and special value formula of Waldspurger in §1. Moreover, we recall basic facts on Galois cohomology and Selmer groups in §2. In §3, we review the theory of vanishing cycles which is used in §4 and §5. In §4, we prepare some fundamental results on the cohomology of Shimura curves. In §5, we show a level raising result for higher weight modular forms and prove a key result to compute the image of CM cycles under the ℓ-adic Abel-Jacobi map. In §6 and §7, we construct a special cohomology classes using CM cycles on Kuga-Sato varieties and give a proof of the main theorem.

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1. Theta elements and the special value formula

In this section, we recall the construction of the theta element and the relation with central values of anticyclotomic L-functions for modular forms following [8, §2, 3 and 4].

Fix an embedding $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and an isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$ for each rational prime p, where $\mathbb{C}_p$ is the p-adic completion of an algebraic closure of $\mathbb{Q}_p$. Let $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/m\mathbb{Z}$ be the finite completion of $\mathbb{Z}$. For $\mathbb{Z}$-algebra A, we denote $A \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ by $\widehat{A}$.

Let K be an imaginary quadratic field with the discriminant $-D_K < 0$ and let $\delta = \sqrt{-D_K}$. Denote $z \mapsto \bar{z}$ for the complex conjugate on K. Define $\boldsymbol{\theta}$ by

$$\boldsymbol{\theta} = \frac{D' + \delta}{2},\ D' = \begin{cases} D_K & \text{if } 2 \nmid D_K, \\ D_K/2 & \text{if } 2 \mid D_K. \end{cases}$$

Fix positive integers N^+ that are only divisible by prime split in K and N^- that are only divisible by primes inert in K. We assume that N^- is the square-free product of an odd number of primes. Let B be the definite quaternion over $\mathbb{Q}$ which is ramified at the prime factors of N^- and the archimedean place. We can regard K as a subalgebra of B. Write T and N for the reduced trace and norm of B respectively. Let $G = B^\times$ be the algebraic group over $\mathbb{Q}$ and let $Z = \mathbb{Q}^\times$ be the center of G. Let $\ell \nmid N^-$ be a rational prime. Let m be a positive integer such that $(m, N^+N^-\ell) = 1$. We choose a basis of $B = K \oplus K \cdot J$ over K such that

- $J^2 = \beta \in \mathbb{Q}^\times$ with $\beta < 0$ and $Jt = \bar{t}J$ for all $t \in K$.
- $\beta \in (\mathbb{Z}_q^\times)^2$ for all $q \mid N^+$ and $\beta \in \mathbb{Z}_q^\times$ for $q|D_K$.

Fix a square root $\sqrt{\beta} \in \overline{\mathbb{Q}}$ of β. We fix an isomorphism $i^{(N^-)} = \prod_{q \nmid N^-} i_q : \widehat{B}^{(N^-)} \cong M_2(\mathbb{A}_f^{(N^-)})$ as follows. For each finite place $q|m\ell N^+$, the isomorphism $i_q : B_q \cong M_2(\mathbb{Q}_q)$ is defined by

$$i_q(\boldsymbol{\theta}) = \begin{pmatrix} \mathrm{T}(\boldsymbol{\theta}) & -\mathrm{N}(\boldsymbol{\theta}) \\ 1 & 0 \end{pmatrix},\ i_q(J) = \sqrt{\beta} \cdot \begin{pmatrix} -1 & \mathrm{T}(\boldsymbol{\theta}) \\ 0 & 1 \end{pmatrix} \quad (\sqrt{\beta} \in \mathbb{Z}_q^\times).$$

For each finite place $q \nmid N^+N^-\ell m$, choose the isomorphism $i_q : B_q := B \otimes_{\mathbb{Q}} \mathbb{Q}_q \cong M_2(\mathbb{Q}_q)$ such that

$$i_q(\mathcal{O}_K \otimes \mathbb{Z}_q) \subset M_2(\mathbb{Z}_q).$$

From now on, we shall identify B_q and $G(\mathbb{Q}_q)$ with $M_2(\mathbb{Q}_q)$ and $\mathrm{GL}_2(\mathbb{Q}_q)$ via i_q for finite $q \nmid N^-$. Finally, we define

$$i_K : B \hookrightarrow M_2(K),\ a + bJ \mapsto i_K(a + bJ) := \begin{pmatrix} a & b\beta \\ \bar{b} & \bar{a} \end{pmatrix} \quad (a, b \in K)$$

and let $i_{\mathbb{C}} : B \to M_2(\mathbb{C})$ be the composition $i_{\mathbb{C}} = \iota_\infty \circ i_K$.

Fix a decomposition $N^+\mathcal{O}_K = \mathfrak{N}^+\overline{\mathfrak{N}^+}$ once and for all. For each finite place q, we define $\varsigma_q \in G(\mathbb{Q}_q)$ as follows:

$$\varsigma_q = \begin{cases} 1 & \text{if } q \nmid N^+m, \\ \delta^{-1}\begin{pmatrix} \boldsymbol{\theta} & \overline{\boldsymbol{\theta}} \\ 1 & 1 \end{pmatrix} & \text{if } q = \mathfrak{q}\bar{\mathfrak{q}} \text{ is split with } \mathfrak{q}|\mathfrak{N}^+, \\ \begin{pmatrix} q^n & 0 \\ 0 & 1 \end{pmatrix} & \text{if } q|m \text{ and } q \text{ is inert in } K \ (n = \mathrm{ord}_q(m)), \\ \begin{pmatrix} 1 & q^{-n} \\ 0 & 1 \end{pmatrix} & \text{if } q|m \text{ and } q \text{ splits in } K \ (n = \mathrm{ord}_q(m)). \end{cases}$$

Define $x : \mathbb{A}_K^\times \to G(\mathbb{A})$ by

$$x_m(a) := a \cdot \varsigma \quad (\varsigma := \prod_q \varsigma_q).$$

This collection $\{x_m(a)\}_{a \in \mathbb{A}_K^\times}$ of points is called Gross points of conductor m associated to K.

Let $\mathcal{O}_{K,m} = \mathbb{Z} + m\mathcal{O}_K$ be the order of K of conductor m. For each positive integer M prime to N^-, we denote by R_M the Eichler order of level M with respect to the isomorphisms $\{i_q : B_q \simeq M_2(\mathbb{Q}_q)\}_{q \nmid N^-}$. Then one can see that the inclusion map $K \hookrightarrow B$ is an optimal embedding of $\mathcal{O}_m$ into the Eichler order $B \cap \varsigma\widehat{R}_M(\varsigma)^{-1}$ (i.e. $(B \cap \varsigma\widehat{R}_M(\varsigma)^{-1}) \cap K = \mathcal{O}_{K,m}$) if $\mathrm{ord}_q(M) \le \mathrm{ord}_q(m)$ for all primes $q|m$.

Let $k \ge 2$ be an even integer. For a ring A, we denote by $L_k(A) = \mathrm{Sym}^{k-2}(A^2)$ the set of homogeneous polynomials in two variables of degree $k-2$ with coefficients in A. We write

$$L_k(A) = \bigoplus_{-\frac{k}{2}<r<\frac{k}{2}} A \cdot \boldsymbol{v}_r \quad (\boldsymbol{v}_r := X^{\frac{k-2}{2}-r}Y^{\frac{k-2}{2}+r}).$$

Also we let $\rho_k : \mathrm{GL}_2(A) \to \mathrm{Aut}_A L_k(A)$ be the unitary representation defined by

$$\rho_k(g)P(X,Y) = \det(g)^{-\frac{k-2}{2}} \cdot P((X,Y)g) \quad (P(X,Y) \in L_k(A)).$$

If A is a $\mathbb{Z}_{(\ell)}$-algebra with $\ell > k-2$, we define a perfect pairing $\langle\ ,\ \rangle_k : L_k(A) \times L_k(A) \to A$ by

$$\langle \sum_i a_i \boldsymbol{v}_i, \sum_j b_j \boldsymbol{v}_j \rangle_k = \sum_{-k/2<r<k/2} a_r b_{k-2-r} \cdot (-1)^{\frac{k-2}{2}+r} \frac{\Gamma(k/2+r)\Gamma(k/2-r)}{\Gamma(k-1)}.$$

For $P, P' \in L_k(A)$, this pairing satisfies

$$\langle \rho_k(g)P, \rho_k(g)P' \rangle_k = \langle P, P' \rangle_k.$$

Via the embedding $i_{\mathbb{C}}$, we obtain a representation

$$\rho_{k,\infty} : G(\mathbb{R}) = (B \otimes_{\mathbb{Q}} \mathbb{R})^\times \xrightarrow{i_{\mathbb{C}}} \mathrm{GL}_2(\mathbb{C}) \to \mathrm{Aut}_{\mathbb{C}} L_k(\mathbb{C}).$$

Then $\mathbb{C} \cdot \boldsymbol{v}_r$ is the eigenspace on which $\rho_{k,\infty}(t)$ acts by $(\bar{t}/t)^r$ for $t \in (K \otimes_{\mathbb{Q}} \mathbb{C})^\times$. If A is a K-algebra and $U \subset G(\mathbb{A}_f)$ is an open compact subgroup, we denote by $\boldsymbol{S}_k^B(U, A)$ be the space of modular forms of weight k defined over A, consisting of functions $f : G(\mathbb{A}_f) \to L_k(A)$ such that

$$f(\alpha g u) = \rho_{k,\infty}(\alpha) f(g) \text{ for all } \alpha \in G(\mathbb{Q}) \text{ and } u \in U.$$

Denote $\boldsymbol{S}_k^B(A) := \varinjlim_U \boldsymbol{S}_k^B(U, A)$. Let $\mathcal{A}(G)$ be the space of automorphic forms on $G(\mathbb{A})$. For $\boldsymbol{v} \in L_k(\mathbb{C})$ and $f \in \boldsymbol{S}_k^B(\mathbb{C})$, we define a function $\Psi(\boldsymbol{v} \otimes f) : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ by

$$\Psi(\boldsymbol{v} \otimes f)(g) := \langle \rho_{k,\infty}(g_\infty)\boldsymbol{v}, f(g_f) \rangle.$$

Then the map $\boldsymbol{v} \otimes f \mapsto \Psi(\boldsymbol{v} \otimes f)$ gives rise to $G(\mathbb{A})$-equivariant morphism $L_k(\mathbb{C}) \otimes \boldsymbol{S}_k^B(\mathbb{C}) \to \mathcal{A}(G)$. Let ω be a unitary Hecke character of $\mathbb{Q}$. We write $\boldsymbol{S}_k^B(U, \omega, \mathbb{C}) = \{f \in \boldsymbol{S}_k^B(U, \mathbb{C}) \mid f(zg) = \omega(z)f(g) \text{ for all } z \in Z(\mathbb{A})\}$. Let $\mathcal{A}_k^B(U, \omega, \mathbb{C})$ be the space of automorphic forms on $G(\mathbb{A})$ of weight k and central character ω, consisting of functions $\Psi(f \otimes \boldsymbol{v}) : G(\mathbb{A}) \to \mathbb{C}$ for $f \in S_k^B(U, \omega, \mathbb{C})$ and $\boldsymbol{v} \in L_k(\mathbb{C})$. For each positive integer M, we put

$$\boldsymbol{S}_k^B(M, \mathbb{C}) = \boldsymbol{S}_k^B(\widehat{R}_M^\times, \mathbf{1}, \mathbb{C}),$$
$$\mathcal{A}_k^B(M, \mathbb{C}) = \mathcal{A}_k^B(\widehat{R}_M^\times, \mathbf{1}, \mathbb{C}),$$

where $\mathbf{1}$ is the trivial character.

Let π be an unitary cuspidal automorphic representation on $\mathrm{GL}_2(\mathbb{A})$ with trivial central character and π' the unitary irreducible cuspidal automorphic representation on $G(\mathbb{A})$ with trivial central character attached to π via Jacquet-Langlands correspondence. Let π'_{fin} denote the finite constituent of π'. Let $R := R_{N^+}$ be an Eichler order of level N^+. The multiplicity one theorem together with our assumptions imply that π'_{fin} can be realized as a unique $G(\mathbb{A}_f)$-submodule $\boldsymbol{S}^B_k(\pi'_{\mathrm{fin}})$ of $\boldsymbol{S}^B_k(\mathbb{C})$ and $\boldsymbol{S}^B_k(N^+,\mathbb{C})[\pi'_{\mathrm{fin}}] := \boldsymbol{S}^B_k(\pi'_{\mathrm{fin}}) \cap \boldsymbol{S}^B_k(N^+,\mathbb{C})$ is one dimensional. We fix a nonzero new form $f_{\pi'} \in \boldsymbol{S}^B_k(N^+,\mathbb{C})[\pi'_{\mathrm{fin}}]$. Define the automorphic form $\varphi_{\pi'} \in \mathcal{A}^B_k(N^+,\mathbb{C})$ by

$$\varphi_{\pi'} := \Psi(\boldsymbol{v}_0^* \otimes f_{\pi'}) \quad (\boldsymbol{v}_0^* = D_K^{\frac{k-2}{2}} \cdot \boldsymbol{v}_0).$$

Define the local Atkin-Lehner element $\tau_q^{N^+} \in G(\mathbb{Q}_q)$ by $\tau_q^{N^+} = J$ for $q|\infty N^-$, $\tau_q^{N^+} = 1$ for finite place $q \nmid N$ and $\tau_q^{N^+} = \begin{pmatrix} 0 & 1 \\ -N^+ & 0 \end{pmatrix}$ if $q|N^+$. Let $\tau^{N^+} := \prod_q \tau_q^{N^+} \in G(\mathbb{A})$. Let $\mathrm{Cl}(R)$ be a set of representatives of $B^\times \backslash \widehat{B}^\times / \widehat{R}^\times \widehat{\mathbb{Q}}^\times$ in $\widehat{B}^\times = G(\mathbb{A}_f)$. Define the inner product of $f_{\pi'}$ by

$$\langle f_{\pi'}, f_{\pi'} \rangle_R := \sum_{g \in \mathrm{Cl}(R)} \frac{1}{\#\Gamma_g} \cdot \langle f_{\pi'}(g), f_{\pi'}(g\tau^{N^+}) \rangle_k \quad (\Gamma_g := (B^\times \cap g\widehat{R}^\times g^{-1}\widehat{\mathbb{Q}}^\times)/\mathbb{Q}^\times).$$

Let $\ell \nmid N^-$ be a rational prime. We recall the description of ℓ-adic modular forms on $B^\times$. Let A be a $\mathcal{O}_{K_{\mathfrak{l}}}$-algebra. For an open compact subgroup $U \subset \widehat{R}^\times$, we define the space of ℓ-adic modular forms of weight k and level U by

$$\mathcal{S}^B_k(U,A) := \left\{ \widehat{f} : \widehat{B} \to L_k(A) \,\middle|\, \widehat{f}(\alpha g u) = \rho_k(u_\ell^{-1})\widehat{f}(g),\ \alpha \in B^\times,\ u \in U\widehat{\mathbb{Q}} \right\}.$$

Also we write $\mathcal{S}^B_k(N^+,A) := \mathcal{S}^B_k(\widehat{R}^\times, A)$. Let $\overline{\lambda}$ and $\mathfrak{l}$ be the primes of $\overline{\mathbb{Q}}$ and K induced by ι_ℓ respectively. We let $i_{K_{\mathfrak{l}}} : B \hookrightarrow \mathrm{M}_2(K_{\mathfrak{l}})$ be the composition $i_{K_{\mathfrak{l}}} := \iota_\ell \circ i_K$. Define $\rho_{k,\ell} : B_\ell^\times \to \mathrm{Aut} L_k(\mathbb{C}_\ell)$ by

$$\rho_{k,\ell}(g) := \rho_k(i_{K_{\mathfrak{l}}}(g)).$$

By definition, $\rho_{k,\ell}$ is compatible with $\rho_{k,\infty}$ in the sense that $\rho_{k,\ell}(g) = \rho_{k,\infty}(g)$ for every $g \in B^\times$, and one can check that

$$\rho_{k,\ell}(g) = \rho_k(\gamma_{\mathfrak{l}} i_\ell(g) \gamma_{\mathfrak{l}}^{-1}), \text{ where } \gamma_{\mathfrak{l}} := \begin{pmatrix} \sqrt{\beta} & -\sqrt{\beta}\boldsymbol{\theta} \\ -1 & \boldsymbol{\theta} \end{pmatrix} \in \mathrm{GL}_2(K_{\mathfrak{l}}).$$

Here $i_\ell : B_\ell \simeq M_2(\mathbb{Q}_\ell)$ is the fixed isomorphism. If ℓ is invertible in A, there is an isomorphism:

$$\boldsymbol{S}^B_k(N^+,A) \cong \mathcal{S}^B_k(N^+,A),\ f \mapsto \widehat{f}(g) := \rho_k(\gamma_{\mathfrak{l}}^{-1})\rho_{k,\ell}(g_\ell^{-1}) f(g).$$

Let $\mathbb{Q}(f)$ be the finite extension of $\mathbb{Q}$ generated by Fourier coefficients of the newform $f = f_\pi \in S_k^{\mathrm{new}}(\Gamma_0(N))$. Let $\mathcal{O} \subset \mathbb{C}_\ell$ be the completion of the ring of integers of $\mathbb{Q}(f)$ with respect to $\lambda = \overline{\lambda} \cap \mathbb{Q}(f)$. Fix a uniformizer λ in $\mathcal{O}$. The $\mathcal{O}$-module $\mathcal{S}^B_k(N^+,\mathcal{O})[\pi'_{\mathrm{fin}}] := \mathcal{S}^B_k(N^+,\mathcal{O}) \cap \mathcal{S}^B_k(N^+,\mathbb{C}_\ell)[\pi'_{\mathrm{fin}}]$ has rank one. We say $f_{\pi'} \in \boldsymbol{S}^B_k(N^+,\mathbb{C})[\pi'_{\mathrm{fin}}]$ is λ-adically normalized if $\widehat{f_{\pi'}}$ is a generator of $\mathcal{S}^B_k(N^+,\mathcal{O})[\pi'_{\mathrm{fin}}]$ over $\mathcal{O}$. This is equivalent to the following condition:

$$\widehat{f_{\pi'}}(g_0) \not\equiv 0 \pmod{\lambda} \text{ for some } g_0 \in G(\mathbb{A}_f).$$

Now we define the theta elements. For a positive integer m, let $\mathcal{G}_m = K^\times \backslash \hat{K}^\times / \widehat{\mathcal{O}}_{K,m}^\times$ be the Picard group of the order $\mathcal{O}_{K,m} = \mathbb{Z} + m\mathcal{O}_K$. We identify $\mathcal{G}_m$ with the Galois group of the ring class field K_m of conductor m over K via geometrically normalized reciprocity law.

Denote by $[\,\cdot\,]_m : \widehat{K}^\times \to \mathcal{G}_m$, $a \mapsto [a]_m$ the natural projection map. We consider the automorphic form $\varphi_{\pi'} = \Psi(\boldsymbol{v}_0^* \otimes f_{\pi'})$. It is easy to see that the function

$$\widehat{\varphi}_{\pi'} : \widehat{K}^\times \to \mathbb{C}, \quad a \mapsto \widehat{\varphi}_{\pi'}(a) := \varphi_{\pi'}(x_m(a))$$

factors through $\mathcal{G}_m$, so we can extend $\widehat{\varphi}_{\pi'}$ linearly to be a function $\widehat{\varphi}_{\pi'} : \mathbb{C}[\mathcal{G}_m] \to \mathbb{C}$. Let $P_m := [1]_m \in \mathcal{G}_m$ be the distinguished Gross point of conductor m. We put

$$\widehat{\varphi}_{\pi'}(\sigma(P_m)) = \varphi_{\pi'}(x_m(a)) \text{ if } \sigma = [a]_m \in \mathcal{G}_m.$$

We define the theta element $\Theta(f_{\pi'}) \in \mathbb{C}[\mathcal{G}_m]$ by

$$\Theta(f_{\pi'}) := \sum_{\sigma \in \mathcal{G}_m} \widehat{\varphi}_{\pi'}(\sigma(P_m)) \cdot \sigma.$$

Then we have the following special value formula.

Proposition 1.1. *Let χ be a character of $\mathcal{G}_m$. Then we have*

$$\chi(\Theta(f_{\pi'})^2) = \Gamma(k/2)^2 \cdot \frac{L(f_\pi/K, \chi, k/2)}{\Omega_{\pi,N^-}} \cdot (-1)^{\frac{k}{2}} \cdot m \cdot D_K^{k-1} \cdot \frac{(\#\mathcal{O}_K/2)^2}{2} \sqrt{-D_K}^{-1} \cdot \chi(\mathfrak{N}^+),$$

where

$$\Omega_{\pi,N^-} = \frac{4^{k-1}\pi^k ||f_\pi||_{\Gamma_0(N)}}{\langle f_{\pi'}, f_{\pi'} \rangle_R}$$

is the λ-normalized period for f.

Proof. This formula is a special case of Hung's result [15, Proposition 5.3]. Also see [8, Proposition 4.3] for the case that χ is an unramified character. □

2. Selmer groups for modular forms

2.1. **Definition of Selmer groups.** First we recall the definition of Selmer groups following Bloch-Kato [5]. Let f be a cuspidal Hecke eigenform of weight k with respect to $\Gamma_0(N)$. Let $\mathbb{Q}(f)$ denote the Hecke field generated by eigenvalues $\{a_q(f)\}$ of Hecke operators $\{T_q\}$. Let λ be the prime of $\mathbb{Q}(f)$ above the prime ℓ induced by the fixed embedding ι_ℓ. Denote $E = \mathbb{Q}(f)_\lambda$. Also we denote the integer ring of E by $\mathcal{O}$ and the uniformizer by λ and write $\mathcal{O}_n = \mathcal{O}/\lambda^n\mathcal{O}$. Then there exist a 2-dimensional Galois representation

$$\rho_f = \rho_{f,\lambda} : G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(E)$$

such that $\det(1 - \rho_f(\mathrm{Frob}_q) \cdot X) = 1 - a_q(f)X + q^{k-1}X^2$ for any prime q satisfying $q \nmid \ell N$. Let V_f be the representation space of $\rho_f \otimes \varepsilon_\ell^{\frac{2-k}{2}}$, where $\varepsilon_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^\times$ is the ℓ-adic cyclotomic character. We choose a $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice T_f in V_f, and denote $A_f = V_f/T_f$. Then there is an exact sequence $0 \to T_f \xrightarrow{i} V_f \xrightarrow{\mathrm{pr}} A_f \to 0$.

For a finite extension $F/\mathbb{Q}_p$, Bloch-Kato [5] defined the finite part of Galois cohomology groups by

$$H^1_f(F, V_f) := \begin{cases} \mathrm{Ker}\left[H^1(F, V_f) \to H^1(F^{ur}, V_f)\right] & \ell \neq p, \\ \mathrm{Ker}\left[H^1(F, V_f) \to H^1(F, V_f \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})\right] & \ell = p, \end{cases}$$

where B_{cris} is the p-adic period ring defined by Fontaine and F^{ur} is the maximal unramified extension of F. Also we denote

$$H^1_f(F, T_f) = i^{-1}(H^1_f(F, V_f))$$

and

$$H^1_f(F, A_f) = \mathrm{Im}\left[H^1_f(F, V_f) \hookrightarrow H^1(F, V_f) \xrightarrow{\mathrm{pr}} H^1(F, A_f)\right].$$

For a number field F, we define the λ-part of the Selmer group of f by

$$\mathrm{Sel}(F, A_f) = \mathrm{Ker}\left[H^1(F, A_f) \to \prod_v \frac{H^1(F_v, A_f)}{H^1_f(F_v, A_f)}\right].$$

We also define

$$H^1_f(F, V_f) = \mathrm{Ker}\left[H^1(F, V_f) \to \prod_v \frac{H^1(F_v, V_f)}{H^1_f(F_v, V_f)}\right].$$

Moreover we set $A_{f,n} = A_f[\lambda^n] = \mathrm{Ker}[A_f \xrightarrow{\lambda^n} A_f]$ and $T_{f,n} = T_f/\lambda^n T_f$. Then there exists a Galois equivariant bilinear pairing $T_f \times T_f \to \mathcal{O}(1)$ such that the induced pairings on $T_{f,n} \cong A_{f,n}$ are non-degenerate for all n. For details, see Nekovář [24, Proposition 3.1].

Proposition 2.1. *The pairing above induces the local Tate pairing*

$$\langle\ ,\ \rangle_v : H^1(F_v, T_f) \times H^1(F_v, A_f) \to H^2(F_v, E/\mathcal{O}(1)) \cong E/\mathcal{O},$$

$$\langle\ ,\ \rangle_v : H^1(F_v, T_{f,n}) \times H^1(F_v, A_{f,n}) \to H^2(F_v, \mathcal{O}_n(1)) \cong \mathcal{O}_n,$$

for each place v of F. The local Tate pairing is perfect and satisfies the following properties.

(1) *The pairing $\langle\ ,\ \rangle_v$ makes $H^1_f(F_v, T_f)$ and $H^1_f(F_v, A_f)$ exact annihilators of each other at any place v.*
(2) *If x and y belong to $H^1(F, A_{f,n})$, then*

$$\sum_v \langle x, y\rangle_v = 0,$$

where the sum is over all places v of F but is a finite sum.

Proof. See Besser [4, Proposition 2.2]. □

Definition 2.2. For each place v, we define $H^1_f(F_v, A_{f,n})$ to be the preimage of $H^1_f(F_v, A_f)$ in $H^1(F_v, A_{f,n})$. Then we let

$$\mathrm{Sel}(F, A_{f,n}) = \mathrm{Ker}\left[H^1(F, A_{f,n}) \to \prod_v \frac{H^1(F_v, A_{f,n})}{H^1_f(F_v, A_{f,n})}\right].$$

Also we define $H^1_f(F_v, T_{f,n})$ to be the image of $H^1_f(F_v, T_f)$ in $H^1(F_v, T_{f,n})$. Moreover we define the singular part of local cohomology group $H^1_{sing}(F_v, T_{f,n})$ to be the quotient

$$H^1_{sing}(F_v, T_{f,n}) = \frac{H^1(F_v, T_{f,n})}{H^1_f(F_v, T_{f,n})}.$$

If v does not divide N, then we have

$$H^1_{sing}(F_v, T_{f,n}) = H^1(F^{ur}, T_{f,n})^{G_{F_v}}.$$

By Proposition 2.1, $H^1_f(F_v, A_{f,n})$ and $H^1_{sing}(F_v, T_{f,n})$ are the Pontryagin dual of each other.

For each prime q and $G_{\mathbb{Q}}$-module M, we denote

$$H^1_f(F_q, M) = \bigoplus_{v|q} H^1_f(F_v, M)$$

and

$$H^1_{sing}(F_q, M) = \bigoplus_{v|q} H^1_{sing}(F_v, M).$$

Lemma 2.3. *Let q be a prime which splits in K. Then $H^1_{sing}(K_{m,q}, T_{f,n}) = 0$ for sufficiently large m.*

Proof. This lemma follows from the same argument of Proof of [3, Lemma 2.4]. □

2.2. **Euler system argument.** Here we give a generalization of the Euler system argument introduced by Bertolini-Darmon [3] to the case of higher weight modular forms.

Definition 2.4. A prime p is said to be n-admissible if

(1) p does not divide $N\ell\,[K_m : K]$.
(2) p is inert in K.
(3) λ does not divide $p^2 - 1$.
(4) λ^n divides $p^{\frac{k}{2}} + p^{\frac{k-2}{2}} - \varepsilon \cdot a_p(f)$, where $\varepsilon = \pm 1$.

Lemma 2.5. *Let p be an n-admissible prime. Then $H^1_f(K_{m,p}, A_{f,n})$ and $H^1_{sing}(K_{m,p}, T_{f,n})$ are both isomorphic to $\mathcal{O}_n[\mathcal{G}_m]$. In particular, the χ-part of these groups are both isomorphic to $\mathcal{O}_n$.*

Proof. This is a direct generalization of [3, Lemma 2.6]. □

Define the map ∂_p to be the composition of the maps

$$H^1(K_m, A_{f,n}) \to H^1(K_{m,p}, A_{f,n}) \to H^1_{sing}(K_{m,p}, A_{f,n}).$$

If $\partial_p(\kappa) = 0$ for $\kappa \in H^1(K_{m,p}, A_{f,n})$ (resp. $H^1(K_{m,p}, T_{f,n})$)), let

$$v_p(\kappa) \in H^1_f(K_{m,p}, A_{f,n}) \text{ (resp. } H^1_f(K_{m,p}, T_{f,n}))$$

denote the natural image of κ under the restriction map ∂_p.

Theorem 2.6 ([9], Theorem 6.3)**.** *Let $s \in H^1(K_m, A_{f,n})$ be a non-zero element. Then there exist infinitely many n-admissible primes p such that $\partial_p(s) = 0$ and $v_p(s) \neq 0$.*

Definition 2.7. For a prime p, we define the compactified Selmer group $H^1_p(K_m, T_{f,n})$ to be

$$H^1_p(K_m, T_{f,n}) = \mathrm{Ker}\left[H^1(K_m, T_{f,n}) \to \prod_{v \nmid p} \frac{H^1(K_{m,v}, T_{f,n})}{H^1_f(K_{m,v}, T_{f,n})}\right].$$

Theorem 2.8. *Let t be a positive integer. Suppose that for all but finitely many n-admissible primes p there exist an element $\kappa_p \in H^1_p(K_m, T_{f,n+t})^\chi$ such that $\lambda^{t-1}\partial_p(\kappa_p) \neq 0$. Then $\lambda^n \mathrm{Sel}(K_m, A_{f,n+t})^\chi = 0$*

Proof. Assume that there exist an element s in $\mathrm{Sel}(K_m, A_{f,n+t})^\chi$ satisfying $\lambda^n s \neq 0$. By Theorem 2.6 and the assumption, we can take a $n+t$-admissible prime p satisfying the following properties simultaneously:

(1) $v_p(\lambda^n s) \neq 0$ and $\partial_p(\lambda^n s) = 0$.
(2) there exist an element $\kappa_p \in H^1_p(K_m, T_{f,n+t})^\chi$ such that $\lambda^{t-1}\partial_p(\kappa_p) \neq 0$.

By the properties of the local Tate pairing, we have

$$\sum_q \langle \lambda^{t-1}\partial_q(\kappa_p), v_q(\lambda^n s)\rangle_q = 0.$$

Since $H^1_f(K_{m,q}, A_{f,n})^\chi$ and $H^1_f(K_{m,q}, T_{f,n})^\chi$ are annihilators for each other, we have

$$\langle \lambda^{t-1}\partial_q(\kappa_p), v_q(\lambda^n s)\rangle_q = 0 \text{ for } q \neq p.$$

Therefore $\langle \lambda^{t-1}\partial_p(\kappa_p), \mathrm{res}_p(\lambda^n s)\rangle_q = 0$ by Proposition 3.1 (2). Since the local Tate pairing is perfect, the assumption $\lambda^{t-1}\partial_p(\kappa_p) \neq 0$ implies $v_p(\lambda^n s) = 0$. This gives a contradiction. □

3. Review of vanishing cycles

In §4 and 5, we will use the theory of vanishing cycles in several important steps. Therefore, in this section we briefly recall the theory of vanishing cycles following the exposition in Rajaei [30].

3.1. **Vanishing cycles.** Let R be a characteristic 0 henselian discrete valuation ring with residue field k of characteristic p. Fix a uniformizer ϖ in R. Denote the fraction field by K and the maximal unramified extension of K by K^{ur}. Let $X \to S = \mathrm{Spec}\, R$ be a proper and generically smooth curve and $\mathscr{F}$ a constructible torsion sheaf on X whose torsion is prime to p. Let $i : X_k \to X$, $j : X_K \to X$, $\bar{i} : X_{\bar{k}} \to X_{\mathcal{O}_{K^{\mathrm{ur}}}}$ and $\bar{j} : X_{\overline{K}} \to X_{\mathcal{O}_{K^{\mathrm{ur}}}}$ be the canonical maps. By the proper base change theorem and the Leray spectral sequence for $\bar{j}$, we have

$$R\Gamma(X_{\overline{K}}, \bar{j}^*\mathscr{F}) = R\Gamma(X_{\mathcal{O}_{K^{\mathrm{ur}}}}, R\bar{j}_*\bar{j}^*\mathscr{F}) \xrightarrow{\simeq} R\Gamma(X_{\bar{k}}, \bar{i}^* R\bar{j}_*\bar{j}^*\mathscr{F}).$$

Then the adjunction morphism gives $\phi : \bar{i}_*\mathscr{F} \to \bar{i}^* R\bar{j}_*\bar{j}^*\mathscr{F}$. We define the vanishing cycles by

$$R\Phi\mathscr{F} := \mathrm{Cone}(\phi),$$

and the nearby cycles by

$$R\Psi\mathscr{F} := \bar{i}^* R\bar{j}_*\bar{j}^*\mathscr{F}.$$

Then we have a distinguished triangle

$$\to i^*\mathscr{F} \to R\Psi\mathscr{F} \to R\Phi\mathscr{F} \xrightarrow{+1}.$$

For $i > 0$, we have $R^i\Phi\mathscr{F} = R^i\Psi\mathscr{F}$. Let Σ be the set of singular points of $X_{\bar{k}}$. Assume that a neighbourhood of each singular point x is (locally) isomorphic to the subscheme of $\mathbb{A}^2_S = S[t_1, t_2]$ with the equation $t_1 t_2 = a_x$ (denote $e_x := v(a_x) > 0$). When the special fiber $X_{\bar{k}}$ is reduced, Deligne [10] proved the sheaves $R^i\Phi\mathscr{F}$

vanish for $i \neq 1$ and $R^1\Phi\mathscr{F}$ is supported at Σ, and the specialization map $H^1(X_{\overline{k}}, \mathscr{F}) \to H^1(X_{\overline{K}}, \mathscr{F})$ is injective. Now we have the specialization sequence

$$\begin{aligned} 0 \longrightarrow H^1(X_{\overline{k}}, i^*\mathscr{F})(1) \longrightarrow H^1(X_{\overline{K}}, \mathscr{F})(1) \xrightarrow{\beta} \bigoplus_{x\in\Sigma}(R^1\Phi\mathscr{F})_x(1) \\ \longrightarrow H^2(X_{\overline{k}}, i^*\mathscr{F})(1) \xrightarrow{\mathrm{sp}(1)} H^2(X_{\overline{K}}, \mathscr{F})(1) \longrightarrow 0. \end{aligned}$$

Then we define the character group for the sheaf $\mathscr{F}$ on X by

$$\mathbb{X}(\mathscr{F}) = \mathrm{Ker}\left[\bigoplus_{x\in\Sigma} R^1\Phi\mathscr{F}_x(1) \to \mathrm{Ker}(\mathrm{sp}(1))\right],$$

so

$$0 \longrightarrow H^1(X_{\overline{k}}, \mathscr{F})(1) \longrightarrow H^1(X_{\overline{K}}, \mathscr{F})(1) \longrightarrow \mathbb{X}(\mathscr{F}) \longrightarrow 0. \tag{3.1}$$

For $x \in \Sigma$, let $(X_{\overline{k}})_x$ be the henselization of $X_{\overline{k}}$ at x and B_x the set of two branches of $X_{\overline{k}}$ at x (i.e. the irreducible components of $(X_{\overline{k}})_x$). For $x \in \Sigma$, we define the module $\mathbb{Z}(x)$ and $\mathbb{Z}'(x)$ by

$$\mathbb{Z}(x) := \mathrm{Coker}\left[\mathbb{Z} \xrightarrow{\mathrm{diag}} \mathbb{Z}^{B_x}\right]$$

and

$$\mathbb{Z}'(x) := \mathrm{Ker}\left[\mathbb{Z}^{B_x} \xrightarrow{\mathrm{sum}} \mathbb{Z}\right].$$

Choose an ordering for B_x for each $x \in \Sigma$ and define a base of $\mathbb{Z}'(x)$ by $\delta'_x := (1, -1)$. Denote the dual basis by $\delta_x \in \mathbb{Z}(x)$. We denote $\Lambda = \mathbb{Z}_\ell$. For $x \in \Sigma$, one has $H^i_x(X_{\overline{k}}, R\Psi\Lambda) = 0$ for $i \neq 1, 2$ and the trace map gives an isomorphism $H^2_x(X_{\overline{k}}, R\Psi\Lambda) \xrightarrow{\simeq} \Lambda(-1)$ and $H^1_x(X_{\overline{k}}, R\Psi\Lambda) \xrightarrow{\simeq} \mathbb{Z}(x) \otimes \Lambda$. Moreover we have $R^1\Phi\Lambda_x \xrightarrow{\simeq} \mathbb{Z}'(x) \otimes \Lambda$. Therefore we have the perfect pairing

$$(R^1\Phi\Lambda)_x \times H^1_x(X_{\overline{k}}, R\Psi\Lambda) \longrightarrow H^2_x(X_{\overline{k}}, R\Psi\Lambda) \xrightarrow{\simeq} \Lambda(-1).$$

This pairing gives the cospecialization map

$$\begin{aligned} 0 \longrightarrow H^0(\widetilde{X}_{\overline{k}}, R\Psi\Lambda) \longrightarrow H^0(\widetilde{X}_{\overline{k}}, i^*\Lambda) \longrightarrow \bigoplus_{x\in\Sigma} H^1_x(X_{\overline{k}}, R\Psi\Lambda) \\ \xrightarrow{\beta'} H^1(X_{\overline{K}}, \Lambda) \longrightarrow H^1(X_{\overline{k}}, i^*\Lambda) \longrightarrow 0, \end{aligned}$$

where $\widetilde{X}_{\overline{k}} \to X_{\overline{k}}$ is the normalization map.

3.2. **Monodromy pairing.** Let ℓ be a prime different from p and let I be the inertia group. We consider the map $t_\ell : I \to \mathbb{Z}_\ell(1)$ which is defined by $\sigma \mapsto \sigma(\varpi^{1/\ell})/\varpi^{1/\ell}$, where ϖ is the uniformizer of R. For $\sigma \in I$ and $x \in \Sigma$, we define the variation map

$$\mathrm{Var}(\sigma)_x : (R^1\Phi\Lambda)_x \to H^1_x(X_{\overline{k}}, R\Psi\Lambda)$$

by $a \mapsto -e_x t_\ell(\sigma)(a\delta_x)\delta_x$, and define the monodromy logarithm

$$N_x : (R^1\Phi\Lambda)_x(1) \to H^1_x(X_{\overline{k}}, R\Psi\Lambda)$$

by $N_x(t_\ell(\sigma)a) = \mathrm{Var}(\sigma)_x(a)$ for $a \in (R^1\Phi\Lambda)_x$ and $\sigma \in I$. Then we have a commutative diagram

$$\begin{array}{ccc} (R^1\Phi\Lambda)_x(1) & \xrightarrow{\simeq} & \mathbb{Z}'(x) \otimes \Lambda \\ \downarrow{\scriptstyle N_x} & & \downarrow{\scriptstyle \phi_x} \\ H^1_x(X_{\overline{k}}, R\Psi\Lambda) & \xrightarrow{\simeq} & \mathbb{Z}(x) \otimes \Lambda, \end{array}$$

where the right vertical map ϕ_x is given by $\delta'_x \mapsto -e_x\delta_x$. Moreover we define the map N by the following diagram:

$$\begin{array}{ccc} H^1(X_{\overline{K}},\Lambda)(1) = H^1(X_{\overline{k}}, R\Psi\Lambda)(1) & \xrightarrow{\beta} & \bigoplus_{x\in\Sigma}(R^1\Phi\mathscr{F})_x(1) \\ \Big\downarrow N & & \Big\downarrow \bigoplus N_x \\ H^1(X_{\overline{K}},\Lambda) = H^1(X_{\overline{k}}, R\Psi\Lambda) & \xleftarrow{\beta'} & \bigoplus_{x\in\Sigma} H^1_x(X_{\overline{k}}, R\Psi\Lambda). \end{array}$$

Then we have an explicit description of the monodromy operator N.

Theorem 3.1 (Picard-Lefschetz formula [11])**.** *Under the notation as above, we have the following formula:*

$$N(t_\ell(\sigma)a) = (\sigma-1)a \quad \text{for } a \in H^1(X_{\overline{K}},\Lambda) \text{ and } \sigma \in I.$$

Let B be the set of irreducible components of $X_{\overline{k}}$. Define the modules $\mathbb{X}$ and $\widehat{\mathbb{X}}$ by the exact sequences

$$0 \longrightarrow \mathbb{X} \longrightarrow \bigoplus_{x\in\Sigma}\mathbb{Z}'(x) \longrightarrow \mathbb{Z}^B \longrightarrow \mathbb{Z} \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^B \longrightarrow \bigoplus_{x\in\Sigma}\mathbb{Z}(x) \longrightarrow \widehat{\mathbb{X}} \longrightarrow 0.$$

Then the monodromy pairing

$$u : \mathbb{X}\otimes\mathbb{X} \to \mathbb{Z}$$

is given by the diagram

$$\begin{array}{ccc} \mathbb{X} & \longrightarrow & \bigoplus_{x\in\Sigma}\mathbb{Z}'(x) \\ \Big\downarrow u_* & & \Big\downarrow \bigoplus_{x\in\Sigma}\phi_x \\ \widehat{\mathbb{X}} & \longleftarrow & \bigoplus_{x\in\Sigma}\mathbb{Z}(x). \end{array}$$

Also we have

$$\mathbb{X}\otimes\Lambda = \mathrm{Im}\left[H^1(X_{\overline{K}},\Lambda)(1) \to \bigoplus_{x\in\Sigma}(R^1\Phi\Lambda)_x(1)\right]$$

and

$$\widehat{\mathbb{X}}\otimes\Lambda = \mathrm{Coker}\left[H^0(\widetilde{X}_{\overline{k}},\Lambda) \to \bigoplus_{x\in\Sigma} H^1_x(X_{\overline{k}}, R\Psi\Lambda)\right].$$

Therefore we obtain the diagram

$$\begin{array}{ccccc} H^1(X_{\overline{K}},\Lambda)(1) & \xrightarrow{c} & \mathbb{X}\otimes\Lambda & \longrightarrow & \bigoplus_{x\in\Sigma}(R^1\Phi\Lambda)_x(1) \\ \Big\downarrow N & & \Big\downarrow u_*\otimes\Lambda & & \Big\downarrow \bigoplus N_x \\ H^1(X_{\overline{K}},\Lambda) & \longleftarrow & \widehat{\mathbb{X}}\otimes\Lambda & \longleftarrow & \bigoplus_{x\in\Sigma} H^1_x(X_{\overline{k}}, R\Psi\Lambda). \end{array}$$

Note that the cokernel of u_* is the group of connected components.

Let $\mathscr{F}$ be a locally constant $\mathbb{Z}_\ell$-sheaf on X. The cospecialization exact sequence is

$$0 \longrightarrow H^0(\widetilde{X}_{\overline{k}}, R\Psi\mathscr{F}) \longrightarrow H^0(\widetilde{X}_{\overline{k}}, i^*\mathscr{F}) \longrightarrow \bigoplus_{x\in\Sigma} H^1_x(X_{\overline{k}}, R\Psi(\mathscr{F}))$$

$$\xrightarrow{\beta'} H^1(X_{\overline{K}},\mathscr{F}) \longrightarrow H^1(X_{\overline{k}}, i^*\mathscr{F}) \longrightarrow 0.$$

Now we define the cocharacter group by

$$\widehat{\mathbb{X}}(\mathscr{F}) := \mathrm{Im}(\beta').$$

Then we have a canonical isomorphism $(R^1\Phi\mathscr{F})_x \simeq (R^1\Phi\Lambda)_x\otimes\mathscr{F}_x$ and a natural map $H^1_x(X_{\overline{k}}, R\Psi\Lambda)\otimes\mathscr{F}_x \to H^1_x(X_{\overline{k}}, R\Psi(\mathscr{F}))$. These maps gives a generalization of monodromy pairing

$$\lambda : \mathbb{X}(\mathscr{F}) \to \widehat{\mathbb{X}}(\mathscr{F})$$

by composition of the maps

$$\begin{array}{ccccc} H^1(X_{\overline{K}}, \mathscr{F})(1) & \longrightarrow & \mathbb{X}(\mathscr{F}) & \longrightarrow & \bigoplus_{x\in\Sigma}(R^1\Phi\mathscr{F})_x(1) \\ \Big\downarrow N & & \Big\downarrow \lambda & & \Big\downarrow \bigoplus N_x\otimes 1 \\ H^1(X_{\overline{K}}, \mathscr{F}) & \longleftarrow & \widehat{\mathbb{X}}(\mathscr{F}) & \longleftarrow & \bigoplus_{x\in\Sigma} H^1_x(X_{\overline{k}}, R\Psi\mathscr{F}). \end{array}$$

Then the monodromy operator N is described by the Picard-Lefschetz formula:

$$N(t_\ell(\sigma)a) = (\sigma - 1)a \quad \text{for } a \in H^1(X_{\overline{K}}, \mathscr{F}) \text{ and } \sigma \in I.$$

We define the component group by

$$\Phi(\mathscr{F}) := \mathrm{Coker}\left[\lambda : \mathbb{X}(\mathscr{F}) \to \widehat{\mathbb{X}}(\mathscr{F})\right].$$

4. Cohomology of Shimura curves

Let M be a positive integer and $M = M^+M^-$ a integer decomposition of M such that $M^- > 1$ is a square-free product of an even number of primes and $(M^+, M^-) = 1$. Let B' be the indefinite quaternion algebra over $\mathbb{Q}$ with discriminant M^-. Fix a prime p dividing M^-. Let B be the definite quaternion algebra over $\mathbb{Q}$ with discriminant M^-/p. We fix a $\mathbb{Q}$-embedding $t' : K \hookrightarrow B'$ and an isomorphism $\varphi_{B,B'} : \widehat{B}^{(p)} \cong \widehat{B}'^{(p)}$. Also we fix an Eichler order R_{M^+} of level M^+ in B.

4.1. **Moduli interpretation of Shimura curves.** Fix a maximal order $\mathcal{O}_{B'}$ of B'.

Let S be a $\mathbb{Z}[1/M]$-scheme. A triple (A, ι, C) is called an abelian surface with quaternionic multiplication with level M^+-structure over S if

(1) A is an abelian scheme over S of relative dimension 2,
(2) $\iota : \mathcal{O}_{B'} \to \mathrm{End}_S(A)$ is an inclusion defining an action of $\mathcal{O}_{B'}$ on A,
(3) C is a subgroup scheme of A of order $(M^+)^2$ which is stable and locally cyclic under the action of $\mathcal{O}_{B'}$.

We denote by $\mathcal{F}_{M^+,M^-}$ the functor from the category of schemes over $\mathbb{Z}[1/M]$ to the category of sets which associates to a scheme S to the set of isomorphism classes of abelian surfaces with quaternionic multiplication with level M^+-structure over S. If M^- is strictly greater than 1, the functor $\mathcal{F}_{M^+,M^-}$ is coarsely representable by a scheme X_{M^+,M^-} over $\mathbb{Z}[1/M]$, with smooth fibers. The scheme X_{M^+,M^-} is a smooth projective geometrically connected curve over $\mathbb{Z}[1/M]$.

Let $d \geq 1$ be an integer relatively prime to M and S a $\mathbb{Z}[1/Md]$-scheme. A quadruple (A, ι, C, ν) is called an abelian surface with quaternionic multiplication by $\mathcal{O}_{B'}$ with level M^+-structure and full level d-structure if (A, ι, C) is a triple as above and

$$\nu : (\mathcal{O}_{B'}/d\mathcal{O}_{B'})_S \to A[d]$$

is an $\mathcal{O}_{B'}$-equivariant isomorphism from the constant group scheme $(\mathcal{O}_{B'}/d\mathcal{O}_{B'})_S$ to the group scheme of d-division points of A.

If $d \geq 4$, we have a fine moduli scheme representing the functor $\mathcal{F}_{M^+,M^-,d}$ from the category of schemes over $\mathbb{Z}[1/Md]$ to the category of sets which associates to a scheme S to the set of isomorphism classes of abelian surfaces with quaternionic multiplication with level M^+-structure over S and full level d-structure. We denote it by $X_{M^+,M^-,d}$. Then the Shimura curve $X_{M^+,M-,d}$ is a smooth projective curve over $\mathbb{Z}[1/Md]$. We have a natural Galois covering

$$\psi : X_{M^+,M^-,d} \to X_{M^+,M^-}$$

with Galois group G_d isomorphic to $G'_d/\{\pm 1\}$, where

$$G'_d := (\mathcal{O}_{B'}/d\mathcal{O}_{B'})^\times \simeq (R'/dR')^\times$$

obtained by forgetting the level d-structure. We set

$$U_d = \left\{ g = (g_v)_v \in \widehat{R}^\times_{M^+} \,\middle|\, g_v \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod d \text{ if } v|d \right\}.$$

The complex uniformization of the Shimura curve $X = X_{M^+,M^-d}$ is given by

$$X(\mathbb{C}) = B'\backslash(\mathbb{C}\setminus\mathbb{R}) \times \widehat{B'}^{\times}/U'_d,$$

where $U'_d = \varphi_{B,B'}(U_d^{(p)})\mathcal{O}_{B'_p}$. For $z' \in \mathbb{C}$ and $b' \in \widehat{B}'^{\times}$, we will denote by $[z', b']_{\mathbb{C}}$ the point on $X(\mathbb{C})$ represented by (z', b').

For a prime q dividing M^+, we can consider a model of $X = X_{M^+,M^-,d}$ over $\mathbb{Z}_q$ using a variant of the moduli functor $\mathcal{F}_{M^+,M^-,d}$. The resulting canonical model $X_{\mathbb{Z}_q}$ is a nodal model, that is,

(1) $X_{\mathbb{Z}_q}$ is proper and flat over $\mathbb{Z}_q$, and its generic fiber is X,
(2) the irreducible components of the special fiber $X_{\mathbb{F}_q}$ are smooth, and the only singularities of $X_{\mathbb{F}_q}$ are ordinary double points.

For the prime p which divides M^-, one may define a model $X_{\mathbb{Z}_p}$ of X over $\mathbb{Z}_p$ via moduli scheme. The model $X_{\mathbb{Z}_p}$ is a nodal model. Moreover, the irreducible components of $X_{\mathbb{F}_p}$ are rational curves.

4.2. **p-adic uniformization of Shimura curves.** Let $\mathcal{H}_p$ be the Drinfeld's p-adic upper half plane. Then $\mathbb{C}_p$-valued points of $\mathcal{H}_p$ are given by $\mathcal{H}_p(\mathbb{C}_p) = \mathbb{P}^1(\mathbb{C}_p)\setminus\mathbb{P}^1(\mathbb{Q}_p)$. Let $\widehat{\mathcal{H}}_p$ be a formal model of $\mathcal{H}_p$ and there is a natural action of $B^\times$ on $\widehat{\mathcal{H}}_p$ via ι_p. Fix a nodal model $X_{\mathbb{Z}_p}$. Write $\widehat{X}$ for the formal completion of $X_{\mathbb{Z}_p}$ along its special fiber. Then $\widehat{X}$ is canonically identified with

$$B^\times\backslash\widehat{\mathcal{H}}_p\widehat{\otimes}_{\mathbb{Z}_p}\widehat{\mathbb{Z}}_p^{\mathrm{ur}} \times \widehat{B}^{(p)\times}/U_d^{(p)},$$

where the action of $b \in B^\times$ on $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ is given by $\mathrm{Frob}_p^{\mathrm{ord}_p N(b)}$([1, Theorem 5.2]). Let X^{an} be the rigid analytification of $X \otimes \mathbb{Q}_p$, then X^{an} is identified with

$$B^\times\backslash\mathcal{H}_p\widehat{\otimes}_{\mathbb{Q}_p}\widehat{\mathbb{Q}}_p^{\mathrm{ur}} \times \widehat{B}^{(p)\times}/U_d^{(p)},$$

and $X(\mathbb{C}_p)$ is identified with

$$B^\times\backslash\mathcal{H}_p(\mathbb{C}_p)\times\widehat{B}^{(p)\times}/U_d^{(p)}.$$

4.3. **Bad reduction of Shimura curves.** Fix a quadratic unramified extension $\mathbb{Q}_{p^2}$ of $\mathbb{Q}_p$. We denote the ring of integer of $\mathbb{Q}_{p^2}$ by $\mathbb{Z}_{p^2}$ and the residue field by $\mathbb{F}_{p^2}$.For $p|M^-$, the dual graph $\mathcal{G}_p(X)$ of the special fiber of $X_{\mathbb{Z}_{p^2}}$ is defined to be the finite graph determined by the following properties.

(1) The set of vertices $\mathcal{V}(\mathcal{G}_p)$ is the set of irreducible components of special fiber $X_{\mathbb{F}_{p^2}}$.
(2) The set of edges $\mathcal{E}(\mathcal{G}_p)$ is the set of singular points of $X_{\mathbb{F}_{p^2}}$.
(3) Two vertices v and v' are joined by an edge if v and v' intersect at the singular point e.

Then the dual graph $\mathcal{G}_p(X)$ is identified with $\mathscr{T}_p/\Gamma$, where $\mathscr{T}_p = (\mathcal{E}_p(\mathscr{T}_p), \mathcal{V}_p(\mathscr{T}_p))$ is the Bruhat-Tits tree for $\mathrm{PGL}_2(\mathbb{Q}_p)$, and the p-adic uniformization of $\widehat{X}_{\mathbb{Z}_p}$ induces the following identifications:

(1) The set $\mathcal{E}(\mathcal{G}_p)$ is identified with the double coset space $B^\times\backslash\widehat{B}^\times/U_d(p)$, where

$$U_d(p) = \left\{ g = (g_v)_v \in U_d \,\middle|\, g_p \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod p \right\}.$$

(2) The set $\mathcal{V}(\mathcal{G}_p)$ is identified with $(B^\times\backslash\widehat{B}^\times/U_d) \times \mathbb{Z}/2\mathbb{Z}$.

4.4. **CM points on Shimura curves.** Let z' be a point in $\mathbb{C}\setminus\mathbb{R}$ fixed by $\iota_\infty(K^\times) \subset \mathrm{GL}_2(\mathbb{R})$. We define the set of CM points unramified at p on the Shimura curve X by

$$\mathrm{CM}_K^{p-\mathrm{ur}}(X) = \left\{ [z', b']_{\mathbb{C}} \,\middle|\, b' \in \widehat{B}'^\times, b'_p = 1 \right\} \subset X(K^{\mathrm{ab}}).$$

Let $\mathrm{rec}_K : \widehat{K}^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ be the geometrically normalized reciprocity map. Then by Shimura's reciprocity law, we have

$$\mathrm{rec}_K(a)[z', b']_{\mathbb{C}} = [z', t'(a)b']_{\mathbb{C}}.$$

Hence one has $\iota_p : \mathrm{CM}_K^{p-\mathrm{ur}}(X) \hookrightarrow X(K_p)$.

4.5. **Ribet's exact sequence for higher weight modular forms.** Let k be a positive even integer. Let $\mathscr{F}_k$ be the lisse ℓ-adic sheaf on the Shimura curve $X_{M^+,M^-,d}$ which is defined in Diamond-Taylor [13, §3]. We will use the sheaf $\mathscr{F} = \mathscr{F}_k(\frac{k-2}{2}) \otimes \mathcal{O}$.

Denote the character group and the cocharacter group associated to the Shimura curve $X_{M^+,M^-,d}$ and the sheaf $\mathscr{F}$ by $\mathbb{X}_p(M^+, M^-, d)$ and $\widehat{\mathbb{X}}_p(M^+, M^-, d)$. Also we denote by $\Phi_p(M^+, M^-, d)$ the component group. Let $\Sigma_p = \Sigma_p(M^+, M^-, d)$ be the set of singular points of the special fiber of $X_{M^+,M^-,d}$ at p.

We fix a prime q dividing M^- such that $q \neq p$. Let $\mathbb{T}$ be the Hecke algebra acting on the character group $\mathbb{X}_p(M^+, M^-, d)$. Let $\mathbb{T}'$ be the Hecke algebra acting on $\mathbb{X}_q(M^+pq, M^-/pq, d)$. Let $\mathbb{T}''$ be the Hecke algebra acting on $\mathbb{X}_q(M^+q, M^-/pq, d)^2$ and let $\widetilde{\mathbb{T}}$ be the polynomial ring with $\mathbb{Z}$-coefficient generated by indeterminates $\widetilde{T}_v$ for $v \nmid Md$ and $\widetilde{U}_v$ for $v|Md$.

Proposition 4.1. *Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal.*

(1) (Ribet's exact sequence) *There is a Hecke equivariant exact sequences*

$$0 \to \widehat{\mathbb{X}}_q(M^+q, M^-/pq, d)^2_{\mathfrak{m}} \to \widehat{\mathbb{X}}_q(M^+pq, M^-/pq, d)_{\mathfrak{m}} \to \widehat{\mathbb{X}}_p(M^+, M^-, d)_{\mathfrak{m}} \to 0$$

and

$$0 \to \mathbb{X}_p(M^+, M^-, d)_{\mathfrak{m}} \to \mathbb{X}_q(M^+pq, M^-/pq, d)_{\mathfrak{m}} \to \mathbb{X}_q(M^+q, M^-/pq, d)^2_{\mathfrak{m}} \to 0.$$

(2) *The action of $U'_p \in \mathbb{T}'$ on $\mathbb{X}_q(M^+q, M^-/pq, d)^2$ is given by $(x, y) \mapsto (T''_p x - p^{-\frac{k-4}{2}} y, p^{k-1}x)$.*

Proof. These results are explained in Rajaei [30, §3.2]. □

The Hecke algebra $\mathbb{T}'$ is isomorphic to the Hecke algebra acting on $S^B_k(U_d, \mathcal{O})$ the space of quaternionic modular forms on B of level U_d. The Hecke algebra $\mathbb{T}$ is isomorphic to the Hecke algebra acting on $S^{B'}_k(U'_d, \mathcal{O})$. Also the Hecke algebra $\mathbb{T}''$ is isomorphic to the Hecke algebra acting on the space of quaternionic modular forms on B' of level U'_d which are old at p.

Lemma 4.2. *There is a canonical map*

$$\omega_p : \mathrm{Ker}\,[\mathrm{sp}(1)] \to \Phi_p(M^+, M^-, d),$$

where $\mathrm{sp}(1) : H^2(X_{M^+,M^-,d} \otimes \overline{\mathbb{F}_{p^2}}, \mathscr{F})(1) \to H^2(X_{M^+,M^-,d} \otimes \overline{\mathbb{Q}_{p^2}}, \mathscr{F})(1)$ *is the specialization map.*

Proof. For $c \in \mathrm{Ker}\,(\mathrm{sp}(1))$, let $\widetilde{c}$ be a lift of c by the map

$$\bigoplus_{x \in \Sigma_p} (R^1\Phi\mathscr{F})_x(1) \to H^2(X_{M^+,M^-,d} \otimes \overline{\mathbb{F}_{p^2}}, \mathscr{F})(1).$$

Then the monodromy pairing induces the map

$$\bigoplus_{x \in \Sigma_p} (R^1\Phi\mathscr{F})_x(1) \to \bigoplus_{x \in \Sigma_p} H^1_x(X_{M^+,M^-,d} \otimes \overline{\mathbb{F}_{p^2}}, R\Psi\mathscr{F}).$$

Also we have a natural surjective map

$$H^1_x(X_{M^+,M^-,d} \otimes \overline{\mathbb{F}_{p^2}}, R\Psi\mathscr{F}) \to \widehat{\mathbb{X}}_p(M^+, M^-, d) \to \Phi_p(M^+, M^-, d).$$

Moreover one can see that the image of $\widetilde{c}$ in the component group does not depend on the choice of lift of c. Then we define $\omega_p(c)$ by the natural image of $\widetilde{c}$. □

Proposition 4.3. *Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal. Then the map ω_p induces a $\widetilde{\mathbb{T}}$-equivariant isomorphism*

$$\overline{\omega}_p : \mathbb{X}_q(M^+q, M^-/pq, d)_{\mathfrak{m}} \times \mathbb{X}_q(M^+q, M^-/pq, d)_{\mathfrak{m}}/((U'_p)^2 - p^{k-2}) \to \Phi_p(M^+, M^-, d)_{\mathfrak{m}}.$$

Proof. Write $\mathbb{X}_p$ for $\mathbb{X}_p(M^+, M^-, d)_{\mathfrak{m}}$, $\mathbb{X}'_q$ for $\mathbb{X}_q(M^+pq, M^-/pq, d)_{\mathfrak{m}}$ and $\mathbb{X}''_q$ for $\mathbb{X}_q(M^+q, M^-/pq, d)_{\mathfrak{m}}$. Let

$$\lambda''_q : \mathbb{X}''_q \times \mathbb{X}''_q \to \widehat{\mathbb{X}}''_q \times \widehat{\mathbb{X}}''_q$$

and

$$\lambda'_q : \mathbb{X}'_q \to \widehat{\mathbb{X}}'_q$$

be the monodromy pairings, hence the cokernel is $\Phi''_q \times \Phi''_q$ and Φ'_q, where $\Phi''_q = \Phi_q(M^+q, M^-/pq)_{\mathfrak{m}}$ and $\Phi'_q = \Phi_q(M^+pq, M^-/pq)_{\mathfrak{m}}$. Let

$$i : \mathbb{X}_p \to \mathbb{X}'_q$$

be the map as in the second exact sequence of Proposition 6.5 (1) and

$$\delta^\vee_* : \widehat{\mathbb{X}}''_q \times \widehat{\mathbb{X}}''_q \to \widehat{\mathbb{X}}'_q/\lambda'_q(i(\mathbb{X}_p)) \tag{4.2}$$

the map obtained by the first exact sequence of Proposition 6.5 (1).Then the cokernel of $\delta^\vee_*$ is $\Phi_p = \Phi_p(M^+, M^-)$. Let

$$j_0 : \mathbb{X}''_q \times \mathbb{X}''_q \to \widehat{\mathbb{X}}'_q/\lambda'_q(i(\mathbb{X}_p)).$$

be the composition of the map λ''_q with $\xi : \widehat{\mathbb{X}}''_q \times \widehat{\mathbb{X}}''_q \to \widehat{\mathbb{X}}'_q$ as in the first exact sequence of Proposition 6.5 (1). Moreover we define the map $\sigma : \mathbb{X}''_q \times \mathbb{X}''_q \to \mathbb{X}''_q \times \mathbb{X}''_q$ by

$$(x, y) \mapsto ((p + 1)x + T''_p y, p^{\frac{k-2}{2}} T_p x + (p + 1)y).$$

One obtains a commutative diagram

$$\begin{CD}
0 @>>> \mathbb{X}''_q \times \mathbb{X}''_q @>{\lambda_q}>> \widehat{\mathbb{X}}''_q \times \widehat{\mathbb{X}}''_q @>>> \Phi''_q \times \Phi''_q @>>> 0\\
@. @VV{\sigma}V @VV{\delta^\vee_*}V @VVV @.\\
0 @>>> \mathbb{X}''_q \times \mathbb{X}''_q @>{j_0}>> \widehat{\mathbb{X}}'_q/\lambda'_q(i(\mathbb{X}_p)) @>>> \Phi'_q @>>> 0.
\end{CD}$$

In fact $\Phi'_q = \Phi''_q = 0$. A direct calculation shows that the composition of the morphism

$$(x, y) \mapsto (-p^{\frac{k-2}{2}} x, T''_p x - p^{-\frac{k-2}{2}} y)$$

of $\mathbb{X}''_q \times \mathbb{X}''_q$ with σ gives the action of $(U'_p)^2 - p^{k-2}$. By the snake lemma, we have the isomorphism

$$\mathbb{X}''_q \times \mathbb{X}''_q/((U'_p)^2 - p^{k-2}) \to \Phi_p(M^+, M^-, d)_{\mathfrak{m}}.$$

□

4.6. **Integral Hodge theory following Jordan-Livné.** In this section, we give a different description of the component groups following Jordan-Livné [20]. Let $X_{\mathbb{Z}_p}$ be the integral model of the Shimura curve $X_{M^+,M^-,d}$ over $\mathbb{Z}_p$ discussed in the beginning of §6. Let X_s be the special fiber of $X^{\mathrm{ur}}_{\mathbb{Z}_p} = X_{\mathbb{Z}_p} \otimes \mathbb{Z}^{\mathrm{ur}}_p$ and X_η the generic fiber. Define

$$C^0(\mathcal{G}_p, \mathscr{F}) := \bigoplus_{y \in I} \mathscr{F}_y \,(\cong H^2(X_s, \mathscr{F})(1))$$

and

$$C^1(\mathcal{G}_p, \mathscr{F}) := \bigoplus_{x \in \Sigma_p} \mathscr{F}_x \left(\cong \bigoplus_{x \in \Sigma_p} (R^1\Phi\mathscr{F})_x(1)\right),$$

where $\mathcal{G}_p = \mathcal{G}_p(X)$ is the dual graph of the special fiber of $X_{\mathbb{Z}_p}$ and I is the set of irreducible components of X_s. We fix an orientation of $\mathcal{G}_p$, that is, a pair of maps $s, t : \mathcal{E}(\mathcal{G}_p) \to \mathcal{V}(\mathcal{G}_p)$ such that $s(e)$ and $t(e)$ are the end of the edge e. Consider the map

$$d : C^0(\mathcal{G}_p, \mathscr{F}) \to C^1(\mathcal{G}_p, \mathscr{F})$$

defined by $(y \mapsto f_y) \mapsto (x \mapsto f_{t(x)} - f_{s(x)})$, where $f_y \in \mathscr{F}_y$ and $f_{t(x)}, f_{s(x)} \in \mathscr{F}_x = H^0_x(s(x), r^*\mathcal{F})(1) = H^0_x(t(x), r^*\mathscr{F})(1)$ (where $r^* : \widetilde{X}_s \to X_s$ is the normalization map). Note that $r^*\mathscr{F}$ is a constant sheaf on $t(x) \cup s(x)$. Then we define the cohomology $H^i(\mathcal{G}_p, \mathscr{F})$ by the exact sequence

$$0 \to H^0(\mathcal{G}_p, \mathscr{F}) \to C^0(\mathcal{G}_p, \mathscr{F}) \xrightarrow{d} C^1(\mathcal{G}_p, \mathscr{F}) \to H^1(\mathcal{G}_p, \mathscr{F}) \to 0.$$

On the other hand, we consider the map

$$\delta : C^1(\mathcal{G}_p, \mathscr{F}) \to C^0(\mathcal{G}_p, \mathscr{F})$$

defined by $(x \mapsto f_x) \mapsto (y \mapsto \sum_{t(x)=y} f_x)$. The Laplacian $\square = \square_i : C^i(\mathcal{G}_p, \mathscr{F}) \to C^i(\mathcal{G}_p, \mathscr{F})$ is defined by $\square_i = d\delta + \delta d$. Hence we have $\square_0 = \delta d$ and $\square_1 = d\delta$. A cochain c is called harmonic if $\square_i c = 0$. Let $\mathbb{H}^i$ be the $\mathcal{O}$-module of all harmonic cochains.

Definition 4.4. We set

$$\Phi'_p(M^+, M^-, d) := H^1(\mathcal{G}_p, \mathscr{F})/\mathbb{H}^1$$

and

$$\Phi''_p(M^+, M^-, d) := \delta C^1(\mathcal{G}_p, \mathscr{F})/\square_0 C^0(\mathcal{G}_p, \mathscr{F}).$$

Remark 4.5. The definition of Φ'' is different from the notation used in Jordan-Livné [20]. The definition of Φ' corresponds to the Grothendieck's description of component group and the definition of Φ'' corresponds to the Raynaud's description of component group.

Lemma 4.6 ([20], Proposition 2.14)**.** *There are canonical identifications*

$$\Phi'_p(M^+, M^-, d) \cong \Phi''_p(M^+, M^-, d) \cong \Phi_p(M^+, M^-, d).$$

In particular, the map ω_p is surjective.

Now for each irreducible component Y we fix a non-singular point P_Y on Y. Let $\tilde{x}$ be a closed point of X_η such that $x = \tilde{x} \mod p$ is not a singular point. We may assume that $x = P_Y$ for some irreducible component Y. Denote

$$H^2_{\tilde{x}}(X_{\overline{\eta}}, \mathscr{F})(1)^0 := \operatorname{Ker}\left[H^2_{\tilde{x}}(X_{\overline{\eta}}, \mathscr{F})(1) \to H^2(X_{\overline{\eta}}, \mathscr{F})(1)\right].$$

Lemma 4.7. *There exists a natural map*

$$H^2_{\tilde{x}}(X_{\overline{\eta}}, \mathscr{F}) \to H^2_x(X_s, R\Psi\mathscr{F}).$$

Proof. Let z be the $\mathbb{Z}_p^{\mathrm{ur}}$-valued point of X determined by $\tilde{x}$. Let $\overline{j}' : \tilde{x} \to z$, $\overline{i}' : x \to z$ and $\overline{i}_{\tilde{x}} : \tilde{x} \to X_{\overline{\eta}}$ be canonical maps. Also define $\overline{i}_x$ and $\overline{i}_z$ similarly. Then by definition one has

$$H^2_{\tilde{x}}(X_{\overline{\eta}}, \overline{j}^*\mathscr{F}) = H^2(\tilde{x}, R\overline{i}^!_{\tilde{x}}\overline{j}^*\mathscr{F})$$

and this is isomorphic to $H^2(x, \overline{i}'^* R\overline{j}'_* R\overline{i}^!_{\tilde{x}}\overline{j}^*\mathscr{F})$. It is known that the last cohomology is isomorphic to $H^2(x, \overline{i}'^* R\overline{i}^!_z R\overline{j}_*\overline{j}^*\mathscr{F})$ (See Fu [14, Proposition 8.4.9]). Therefore using adjunction morphism we have a natural map

$$H^2(x, \overline{i}'^* R\overline{i}^!_z R\overline{j}_*\overline{j}^*\mathscr{F}) \to H^2(x, R\overline{i}^!_x\overline{i}^* R\overline{j}_*\overline{j}^*\mathscr{F}) = H^2_x(X_s, \overline{i}^* R\overline{j}_*\overline{j}^*\mathscr{F}).$$

$\square$

We define the reduction map $\mathrm{red}_p : H^2_{\tilde{x}}(X_{\overline{\eta}}, \mathscr{F})(1)^0 \to H^2(X_s, i^*\mathscr{F})(1)$ by the composition of the maps

$$\begin{aligned} H^2_{\tilde{x}}(X_{\overline{\eta}}, \mathscr{F})(1)^0 &\longrightarrow H^2_x(X_s, R\Psi\mathscr{F})(1) \\ &\xrightarrow{\simeq} H^2_x(X_s, i^*\mathscr{F})(1) \\ &\hookrightarrow \bigoplus_{Y \in I} H^2_{P_Y}(X_s, i^*\mathscr{F})(1) \\ &\xrightarrow{\simeq} H^2(X_s, i^*\mathscr{F})(1), \end{aligned}$$

where the first map is obtained by the above lemma and the second map is the inverse of the specialization map

$$\mathrm{sp}(1)_x : H^2_x(X_s, i^*\mathscr{F})(1) \to H^2_x(X_s, R\Psi\mathscr{F})(1)$$

(since x is a smooth point, $\mathrm{sp}(1)_x$ is an isomorphism). Then the image of the reduction map is contained in the kernel of the specialization map

$$\mathrm{sp}(1) : H^2(X_s, i^*\mathscr{F})(1) \to H^2(X_s, R\Psi\mathscr{F})(1).$$

Using the identification of component groups, we define the map

$$d_p \; : \; H^2_x(X_{\overline{\eta}}, \mathscr{F})(1)^0 \to \Phi_p(M^+, M^-, d)$$

by the composition of the maps

$$H^2_x(X_{\overline{\eta}}, \mathscr{F})(1)^0 \xrightarrow{\mathrm{red}_p} \operatorname{Ker}[\mathrm{sp}(1)] \xrightarrow{\simeq} \delta C^1(\mathcal{G}_p, \mathscr{F}) \to \Phi''_p(M^+, M^-, d) \cong \Phi_p(M^+, M^-, d).$$

Combining these facts, we have the following proposition.

Proposition 4.8. *For $c \in H^2_{\tilde{x}}(X_{\overline{\eta}}, \mathscr{F})(1)$, we have $d_p(c) = \omega_p(\mathrm{red}_p(c))$.*

5. Level raising of modular forms

In this section, we prove a level raising result for modular forms on quaternion algebras.

5.1. **A freeness result on the space of modular forms.** Let N be a positive integer and $N = N^+N^-$ a integer factorization of N, where N^- is a square-free product of an odd number of primes. Let $\widehat{f} : B^\times\backslash\widehat{B}^\times \to L_{k-2}(\mathcal{O})$ be a λ-normalized ℓ-adic modular form corresponding to f via Jacquet-Langlands correspondence. Let q be a prime number dividing N^- and p a prime number which does not divide N. Let B' be the indefinite quaternion algebra over $\mathbb{Q}$ with discriminant pN^-. Choose a positive integer d such that $(d, Np) = 1$ and $\ell \nmid \#(\mathcal{O}_{B'}/d\mathcal{O}_{B'})^\times$. Write $\mathbb{T}^B_k(N^+)$ (resp. $\mathbb{T}^{B'}_k(N^+)$) for the Hecke algebra acting on the space of ℓ-adic modular forms $\mathcal{S}^B_k(N^+, \mathcal{O})$ (resp. $\mathcal{S}^{B'}_k(N^+, \mathcal{O})$). Set $\mathbb{T} = \mathbb{T}^B_k(N^+)_{\mathcal{O}}$ and $\mathbb{T}^{[p]} = \mathbb{T}^{B'}_k(N^+)_{\mathcal{O}}$. We denote by t_v and u_v (resp. T_v and U_v) the Hecke operators in $\mathbb{T}$ (resp. $\mathbb{T}^{[p]}$). The modular form $\widehat{f}$ yields a surjective homomorphism

$$\lambda_f : \mathbb{T} \to \mathcal{O}_n.$$

We write $\mathcal{I}_f$ for the kernel of λ_f, and $\mathfrak{m}$ for the unique maximal ideal of $\mathbb{T}$ containing $\mathcal{I}_f$.

Proposition 5.1. *Assume that the residual Galois representation $\overline{\rho}_f$ satisfies* (CR$^+$). *Then $\mathcal{S}^B_k(N^+, \mathcal{O})_{\mathfrak{m}}$ is a cyclic $\mathbb{T}_{\mathfrak{m}}$-module.*

Proof. Since this proposition follows from the same argument with [8, Proof of Proposition 6.8] and [33, §2 and §3], we only give a sketch of the proof. Let M^+ be an integer such that $(M^+, N^-) = 1$ and let $M = M^+N^-$. Write $\mathcal{S}(M) = \mathcal{S}^B_k(M^+, \mathcal{O})$. Let $\mathbb{T}$ be the Hecke algebra generated over $\mathcal{O}$ by Hecke operators T_q for $q \nmid M$ and U_q for $q \mid M$ in $\mathrm{End}_{\mathcal{O}}\,\mathcal{S}(M)$. Let $\lambda_{\pi'} : \mathbb{T}(N) \to \mathcal{O}$ be the $\mathcal{O}$-algebra homomorphim induced by π'. We denote by$N(\overline{\rho}_f)$ the Artin conductor of $\overline{\rho}_f$. Let N_1^- be the product of prime factors of $N(\overline{\rho}_f)$. We set $N_\emptyset = N(\overline{\rho}_f)N_1^-$. By the level lowering and raising, there exists a modular lifting $\lambda_\emptyset : \mathbb{T} \to \mathcal{O}$ such that $\lambda_\emptyset(T_q) = \lambda_{\pi'}(T_q) \mod \mathfrak{m}_{\mathcal{O}}$ for all $q \nmid N$. We write

$$N = N_\emptyset \prod_q q^{m_q}.$$

Let Σ be a set of prime factors of $N/N_\emptyset$ and set $N_\Sigma = N_\emptyset \prod_{q\in\Sigma} q^{m_q}$. Let $\mathfrak{m}_\Sigma$ be the maximal ideal of $\mathbb{T}(N_\Sigma)$ generated by $\mathfrak{m}_{\mathcal{O}}, T_q - \lambda_\emptyset(T_q)$ for $q \nmid N_\Sigma$ and $U_q - \lambda_{\pi'}(U_q)$ for $q \mid N_\Sigma$. Let $\mathbb{T}_\Sigma = \mathbb{T}(N_\Sigma)_{\mathfrak{m}_\Sigma}$ be the localization at $\mathfrak{m}_\Sigma$. Similarly, we denote the localization of $\mathcal{S}(N_\Sigma)$ at $\mathfrak{m}_\Sigma$ by $\mathcal{S}_\Sigma$. By [8, Lemma 6.3], we have a surjection $\mathbb{T}_\Sigma \twoheadrightarrow \mathbb{T}_\emptyset$. Let $\lambda_\Sigma : \mathbb{T}_\Sigma \to \mathbb{T}_\emptyset \xrightarrow{\lambda_\emptyset} \mathcal{O}$ be the composition and I_{λ_Σ} the kernel of λ_Σ. Set $\mathcal{S}_\Sigma[\lambda_\Sigma] = \{x \in \mathcal{S}_\Sigma \mid I_{\lambda_\Sigma}x = 0\}$ and $\mathcal{S}_\Sigma[\lambda_\Sigma]^\perp = \{x \in \mathcal{S}_\Sigma \otimes_{\mathcal{O}} E \mid \langle x, y\rangle_{R_\Sigma} = 1 \text{ for all } y \in \mathcal{S}_\Sigma[\lambda_\Sigma]\}$, where $R_\Sigma = R_{N_\Sigma/N^-}$ is an Eichler order of level N_Σ/N^-. Then $\mathcal{S}_\Sigma[\lambda_\Sigma]^\perp \supset \mathcal{S}_\Sigma[\lambda_\Sigma]$. We define the congruence module of λ_Σ by $C(N_\Sigma) = \mathcal{S}_\Sigma[\lambda_\Sigma]^\perp/\mathcal{S}_\Sigma[\lambda_\Sigma]$ and the congruence ideal of λ_Σ by $\eta_\Sigma = \lambda_\Sigma(\mathrm{Ann}_{\mathbb{T}_\Sigma}(I_{\lambda_\Sigma}))$.

Let $\mathcal{MF}_{\mathbb{Q}_\ell,\mathcal{O},k}$ denote the abelian category whose objects are finite length $\mathcal{O}$-modules D together with a distinguished submodule D^0 and $\mathrm{Frob}_{\mathbb{Q}_\ell} \otimes 1$-semilinear maps $\varphi_{1-k} : D \to D$ and $\varphi_0 : D^0 \to D$ such that

- $\varphi_{1-k}|_{D^0} = \ell^{k-1}\varphi_0$ and
- $\mathrm{Im}\,\varphi_{1-k} + \mathrm{Im}\,\varphi_0 = D$.

Then there is a fully faithful, $\mathbb{Z}_p$-length preserving, $\mathcal{O}$-additive, contravariant functor $\mathbb{M}$ from $\mathcal{MF}_{\mathbb{Q}_p,\mathcal{O},k}$ to the category of continuous $\mathcal{O}[\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)]$-modules with essential image closed under the formation of sub-objects.

Consider the functor $\mathscr{D}_\Sigma$ from the category of local Noetherian complete $\mathcal{O}$-algebra with the residue field $k_{\mathcal{O}} = \mathcal{O}/\mathfrak{m}_{\mathcal{O}}$ to the category of sets which sends A with the maximal ideal $\mathfrak{m}_A$ to the isomorphism class of deformations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(A)$ of $\overline{\rho}_f$ satisfying

(1) $\det\rho = \varepsilon_\ell$, where $\varepsilon_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$ is the ℓ-adic cyclotomic character,
(2) ρ is minimally ramified,
(3) for each finite length quotient A/I of A the $\mathcal{O}[\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)]$-module $(A/I)^2$ is isomorphic to $\mathbb{M}(D)$ for some object D of $\mathcal{MF}_{\mathbb{Q}_p,\mathcal{O},k}$,
(4) for $q||N_\Sigma/N_\emptyset$, there exists a unramified charcter $\delta_q : \mathrm{Gal}(\overline{\mathbb{Q}_q}/\mathbb{Q}_q) \to A^\times$ such that

$$\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}_q}/\mathbb{Q}_q)} \sim \begin{pmatrix} \delta_q^{-1}\varepsilon_\ell & * \\ 0 & \delta_q \end{pmatrix} \quad \text{and } \delta_q(\mathrm{Frob}_q) \equiv 1 \mod \mathfrak{m}_A, \text{ and}$$

(5) if $q \mid N_1^-$, then $\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}_q}/\mathbb{Q}_q)}$ satisfies

$$\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}_q}/\mathbb{Q}_q)} \sim \begin{pmatrix} \pm\varepsilon_\ell & * \\ 0 & \pm 1 \end{pmatrix} \quad \text{with } * \in \mathfrak{m}_A.$$

Under the assumption (CR$^+$), it is known that $\mathscr{D}_\Sigma$ is represented by the universal deformation

$$\rho_{R_\Sigma} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(R_\Sigma).$$

Then the universality of R_Σ gives rise to surjections of $\mathcal{O}$-algebras $R_\Sigma \twoheadrightarrow R_\emptyset$ and $R_\Sigma \twoheadrightarrow \mathbb{T}_\Sigma$ by [8, Lemma 6.5] and [33, Lemma 2.1]. Let $\wp_\Sigma$ be the kernel of the $\mathcal{O}$-algebra homomorphism

$$R_\Sigma \to R_\emptyset \to \mathbb{T}_\emptyset \xrightarrow{\lambda_\emptyset} \mathcal{O}.$$

By the Taylor-Wiles argument in [33, §2], we deduce that $\mathcal{S}(N_\emptyset)_{\mathfrak{m}_\emptyset}$ is a free $\mathbb{T}_\emptyset$-module of rank one and

$$\#(\wp_\emptyset/\wp_\emptyset^2) = \#C(N_\emptyset) = \#(\mathcal{O}/\eta_\emptyset).$$

Using the argument in [33, §3], we have

$$\#(\wp_{Q_2}/\wp_{Q_2}^2) = \#C(N_{Q_2}) = \#(\mathcal{O}/\eta_{Q_2}),$$

where Q_2 is the set of prime factors $q \mid N/N_\emptyset$ with $m_q = 2$. By [8, Lemma 6.4] and [8, Corollary 6.7], the above equality implies

$$\#(\wp_\Sigma/\wp_\Sigma^2) \mid \#C(N_\Sigma) \mid \#(\mathcal{O}/\eta_\Sigma).$$

Then the proposition follows from [12, Theorem 2.4]. □

Proposition 5.2. *Let $\psi_f : \mathcal{S}_k^B(N^+, \mathcal{O}) \twoheadrightarrow \mathcal{O}$ be the map defined by $h \mapsto \psi_f(h) := \langle \widehat{f}, h\rangle_R$, where $R = R_{N^+}$. Then ψ_f induces an isomorphism*

$$\psi_f : \mathcal{S}_k^B(N^+, \mathcal{O})/\mathcal{I}_f \xrightarrow{\simeq} \mathcal{O}_n.$$

Proof. By Proposition 5.1, $\mathcal{S}_k^B(N^+, \mathcal{O})_\mathfrak{m}$ is a cyclic $\mathbb{T}_\mathfrak{m}$-module. Hence $\mathcal{S}_k^B(N^+, \mathcal{O})/\mathcal{I}_f$ is generated by a modular form g. Since ψ_g is surjective and Hecke operators in $\mathbb{T}$ are self-adjoint with respect to the pairing $\langle\ ,\ \rangle_R$, we have that $\psi_f(g) = \langle f, g\rangle_R \in \mathcal{O}_n^\times$ and the annihilator of g in $\mathbb{T}$ is $\mathcal{I}_f$. Therefore we have an isomorphism $\mathcal{S}_k^B(N^+, \mathcal{O})/\mathcal{I}_f \cong \mathbb{T}/\mathcal{I}_f \cong \mathcal{O}_n$. □

5.2. **Level raising.**

Theorem 5.3. *Let p be an n-admissible prime. Assume that the residual Galois representation $\overline{\rho}_f$ satisfies* (CR$^+$). *Then*

(1) *There exists a surjective homomorphism*

$$\lambda_f^{[p]} : \mathbb{T}^{[p]} \to \mathcal{O}_n$$

such that $\lambda_f^{[p]}(T_q) = \lambda_f(t_q)$ for all $q \nmid Np$, $\lambda_f^{[p]}(U_q) = \lambda_f(u_q)$ for all $q \mid N$, and $\lambda_f^{[p]}(U_p) = \varepsilon \cdot p^{\frac{k-2}{2}}$, where $\varepsilon = \pm 1$ is such that λ^n divides $p^{\frac{k}{2}} + p^{\frac{k-2}{2}} - \varepsilon \cdot \lambda_f(t_p)$.

(2) *Let $\mathcal{I}_f^{[p]} \subset \mathbb{T}^{[p]}$ denote the kernel of the homomorphism $\lambda_f^{[p]}$ and $\Phi_p(N^+, N^-p)$ is the component group associated to the Shimura curve X_{N^+,N^-p} and the lisse sheaf $\mathscr{F}$. Then there is a group isomorphism*

$$\Phi_p(N^+, N^-p)/\mathcal{I}_f^{[p]} \cong \mathcal{S}_k^B(N^+, \mathcal{O})/\mathcal{I}_f \overset{\psi_f}{\cong} \mathcal{O}_n.$$

Proof. Let B' be the indefinite quaternion algebra over $\mathbb{Q}$ of discriminant N^-/q and R'_{N^+q} an Eichler order of level N^+q. Denote the Shimura curve associated to U'_d by $X_{U'_d}$. Also we write the character group for the Shimura curve $X_{U'_d}$ and the lisse ℓ-adic sheaf $\mathfrak{F}$ at q by $\mathbb{X}_q(U'_d)$. Let $\Sigma_q(U'_d)$ be the set of singular points on the special fiber of $X_{U'_d}$. Moreover since $\Sigma_q(U'_d)$ is identified with $B^\times \backslash \widehat{B}^\times / U_d$, we obtain the identification

$$\bigoplus_{x \in \Sigma_q(U'_d)} (R^1\Phi\mathfrak{F})_x \cong \bigoplus_{x \in \Sigma_q(U'_d)} L_k(\mathcal{O}) \cong S_k^B(U_d, \mathcal{O}).$$

Taking $\widehat{R}_{N^+}^{\times}/U_d$-invariant part, we obtain the Hecke-equivariant isomorphism

$$\bigoplus_{x\in\Sigma_q}(R^1\Phi\mathfrak{F})_x \cong S_k^B(N^+,\mathcal{O}),$$

where Σ_q is the set of singular points on the special fiber of a model of $X_{N^+q,N^-/q}$. By [30, Proposition 5], we have

$$\mathbb{X}_q(N^+q,N^-/q)_{\mathfrak{m}} \cong \left(\bigoplus_{x\in\Sigma_q}(R^1\Phi\mathfrak{F})_x\right)_{\mathfrak{m}}.$$

Therefore by Proposition 5.1 one obtains the isomorphism

$$\mathbb{X}_q(N^+q,N^-/q)^2/\mathcal{I}_f \simeq \mathcal{O}_n^2.$$

We denote by T_v' and U_v' the Hecke operators in $\mathbb{T}^{[p]}$. There is an action of $\mathbb{T}^{[p]}$ on $\mathbb{X}_q(N^+q,N^-/q)^2$ induced by t_v for $v\nmid Np$ and u_v for $v\mid N$ and the Hecke operator U_p' acts via the formula

$$(x,y)\mapsto(T_p''x-p^{-\frac{k-4}{2}}y,p^{k-1}x).$$

Since p is n-admissible, the action of t_p modulo $\mathcal{I}_f$ is given by $\varepsilon\cdot(p^{\frac{k}{2}}+p^{\frac{k-2}{2}})$. Then the determinant of $U_p'+\varepsilon\cdot p^{\frac{k-2}{2}}$ is $2p^k(1+p)$. Hence $U_p'+\varepsilon\cdot p^{\frac{k-2}{2}}$ is invertible on $\mathbb{X}_q(N^+q,N^-/q)^2/\mathcal{I}_f$. These facts implies the isomorphism

$$\mathbb{X}_q(N^+q,N^-/q)^2/\langle\mathcal{I}_f,U_p'-\varepsilon\cdot p^{\frac{k-2}{2}}\rangle \simeq \mathbb{X}_q(N^+q,N^-/q)^2/\langle\mathcal{I}_f,(U_p')^2-p^{k-2}\rangle \simeq \mathcal{O}_n.$$

Thus, the action of $\mathbb{T}^{[p]}$ on $\mathbb{X}_q(N^+q,N^-/q)^2/\langle\mathcal{I}_f,U_p'-\varepsilon\cdot p^{\frac{k-2}{2}}\rangle$ is given via a surjective homomorphism

$$\lambda_f':\mathbb{T}^{[p]}\to\mathcal{O}_n.$$

Denote the kernel of λ_f' by $\mathcal{I}_f'$. Then Proposition 6.5 and the residual irreducibility of $\mathfrak{m}$ implies the existence of an isomorphism

$$\Phi_p(N^+,N^-p)/\mathcal{I}_f' \simeq \mathbb{X}_q(N^+q,N^-/q)^2/\langle\mathcal{I}_f,(U_p')^2-p^{k-2}\rangle.$$

This shows that λ_f' factors through $\mathbb{T}$ which gives $\lambda_f^{[p]}$ and $\Phi_p(N^+,N^-p)/\mathcal{I}_f^{[p]}$ is isomorphic to $\mathcal{O}_n$. Let $\mathfrak{m}^{[p]}$ be the maximal ideal of $\mathbb{T}^{[p]}$ containing $\mathcal{I}_f^{[p]}$. The embedding $\mathcal{S}_k^B(N^+,\mathcal{O})_{\mathfrak{m}^{[p]}}\hookrightarrow\mathcal{S}_k^B(N^+,\mathcal{O})_{\mathfrak{m}^{[p]}}^{\oplus2}$ given by $x\mapsto(x,0)$ induces an isomorphism

$$\mathcal{S}_k^B(N^+,\mathcal{O})_{\mathfrak{m}^{[p]}}/(\varepsilon T_p-p^{\frac{k}{2}}-p^{\frac{k-2}{2}}) \cong \mathcal{S}_k^B(N^+,\mathcal{O})_{\mathfrak{m}^{[p]}}^{\oplus2}/(U_p'-\varepsilon p^{\frac{k-2}{2}}).$$

Therefore we have

$$\Phi_p(N^+,N^-p)/\mathcal{I}_f^{[p]} \cong \mathcal{S}_k^B(N^+,\mathcal{O})/\mathcal{I}_f \overset{\psi_f}{\cong} \mathcal{O}_n.$$

□

Write $X_d^{[p]}$ for the Shimura curve $X_{N^+,N^-p,d}$, $\mathbb{X}_{d,p}$ for $\mathbb{X}_p(N^+,N^-p,d)$, $\widehat{\mathbb{X}}_{d,p}$ for $\widehat{\mathbb{X}}_p(N^+,N^-p,d)$ and $\Phi_{d,p}$ for $\Phi_p(N^+,N^-p,d)$. Also write $X^{[p]}$ for the Shimura curve X_{N^+,N^-p}, $\mathbb{X}_p$ for $\mathbb{X}_p(N^+,N^-p)$, $\widehat{\mathbb{X}}_p$ for $\widehat{\mathbb{X}}_p(N^+,N^-p)$ and Φ_p for $\Phi_p(N^+,N^-p)$.

Proposition 5.4. *Under the assumption* (CR$^+$), *the Galois representations* $H^1(X^{[p]}\otimes\overline{\mathbb{Q}},\mathscr{F})(1)/\mathcal{I}_f^{[p]}$ *and* $T_{f,n}$ *are isomorphic.*

Proof. Let $\mathfrak{m}_f^{[p]}$ be the maximal ideal containing $\mathcal{I}_f^{[p]}$. Then $\mathbb{T}^{[p]}/\mathfrak{m}_f^{[p]}$ is isomorphic to $\mathcal{O}_1=\mathbb{F}_\lambda$. First we will show that $H^1(X^{[p]}\otimes\overline{\mathbb{Q}},\mathscr{F})(1)/\mathfrak{m}_f^{[p]}$ is isomorphic to $T_{f,1}$.

By (3.1) and the fact $H^1(X^{[p]}\otimes\overline{\mathbb{F}_{p^2}},\mathscr{F})\cong\widehat{\mathbb{X}}_p$ (see Rajaei [30, p.52 (3.5)]), one obtains an exact sequence

$$0\to(\widehat{\mathbb{X}}_p/\mathfrak{m}_f^{[p]})\otimes\mu_\lambda\to H^1(X^{[p]}\otimes\overline{\mathbb{Q}_{p^2}},\mathscr{F})/\mathfrak{m}_f^{[p]}\otimes\mu_\lambda\to\mathbb{X}_p/\mathfrak{m}_f^{[p]}\to0, \tag{5.3}$$

where $\mu_\lambda=\mathbb{Z}_p(1)\otimes\mathcal{O}/\lambda\mathcal{O}$. Taking the Galois cohomology over $\mathbb{Q}_{p^2}$, we have the exact sequence

$$\mathbb{X}_p/\mathfrak{m}_f^{[p]}\to H^1(\mathbb{Q}_{p^2},(\widehat{\mathbb{X}}_p/\mathfrak{m}_f^{[p]})\otimes\mu_\lambda)\to H^1(\mathbb{Q}_{p^2},H^1(X^{[p]}\otimes\overline{\mathbb{Q}_{p^2}},\mathscr{F})/\mathfrak{m}_f^{[p]}\otimes\mu_\lambda).$$

Since p is an admissible prime, λ does not divide p^2-1, hence we have the identification

$$H^1(\mathbb{Q}_{p^2}, \widehat{\mathbb{X}}_p/\mathfrak{m}_{f_p} \otimes \mu_\lambda) \cong \widehat{\mathbb{X}}_p/\mathfrak{m}_f^{[p]}.$$

By the main theorem of [6] and Eichler-Shimura relation, $H^1(X^{[p]} \otimes \overline{\mathbb{Q}}, \mathscr{F})(1)/\mathfrak{m}_f^{[p]}$ is semisimple over $\mathbb{F}[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$, we have that

$$H^1(X^{[p]} \otimes \overline{\mathbb{Q}}, \mathscr{F})(1)/\mathfrak{m}_f^{[p]} \cong (T_{f,1})^r$$

for some $r \geq 1$. Therefore $H^1(\mathbb{Q}_{p^2}, H^1(X^{[p]} \otimes \overline{\mathbb{Q}_{p^2}}, \mathscr{F})(1)/\mathfrak{m}_f^{[p]})$ is isomorphic to $H^1(\mathbb{Q}_{p^2}, T_{f,1})^r$. By Lemma 2.5, the $\mathbb{F}$-vector space $H^1(\mathbb{Q}_{p^2}, T_{f,1})^r$ is $2r$-dimensional. We claim that

$$\dim_{\mathbb{F}} \mathbb{X}_p/\mathfrak{m}_f^{[p]} \geq r. \tag{5.4}$$

To see this, assume that $\dim_{\mathbb{F}} \mathbb{X}_p/\mathfrak{m}_f^{[p]} \leq r-1$. Then we have $\dim_{\mathbb{F}} \widehat{\mathbb{X}}_p/\mathfrak{m}_f^{[p]} \geq r+1$ by the exact sequence (5.3), which implies $\dim_{\mathbb{F}} \Phi_p/\mathfrak{m}_f^{[p]} \geq 2$ by the definition of the component group. This gives a contradiction.

By the Picard-Lefschetz formula, the monodromy operator N is described as $N(a \otimes t_\ell(\sigma)) = \sigma(a) - a$ for all $a \in H^1(X^{[p]} \otimes \overline{\mathbb{Q}_{p^2}}, \mathscr{F})(1)/\mathfrak{m}_f^{[p]}$ and $\sigma \in I$. One notices that the monodromy operator N acts on each piece $T_{f,1}$, thus N defines the map $N : T_{f,1}(-1) \otimes \mu_\lambda \to T_{f,1}(-1)$.

Lemma 5.5. *The map $N : T_{f,1}(-1) \otimes \mu_\lambda \to T_{f,1}(-1)$ is the zero map. Equivalently, the monodromy pairing is the zero map. In particular, $\widehat{\mathbb{X}}_p/\mathfrak{m}_f^{[p]}$ is isomorphic to $\Phi_p/\mathfrak{m}_f^{[p]}$.*

Proof. If N is non-trivial, we have the inequality

$$\dim_{\mathbb{F}} \mathrm{Im}\left[N : H^1(\mathbb{Q}_{p^2}, T_{f,1}(-1))^r \to H^1(\mathbb{Q}_{p^2}, T_{f,1})^r\right] \geq r.$$

The definition of the monodromy pairing implies

$$\mathrm{Im}(N) = \mathrm{Im}\left[\lambda_p : \mathbb{X}_p/\mathfrak{m}_f^{[p]} \to \widehat{\mathbb{X}}_p/\mathfrak{m}_f^{[p]}\right],$$

where λ_p is the monodromy pairing and its cokernel is the component group $\Phi_p/\mathfrak{m}_f^{[p]}$. Since

$$\dim_{\mathbb{F}} \widehat{\mathbb{X}}_p/\mathfrak{m}_f^{[p]} - \dim_{\mathbb{F}} \mathrm{Im}(N) = \dim_{\mathbb{F}} \Phi_p/\mathfrak{m}_f^{[p]} = 1,$$

we have the inequality $\dim_{\mathbb{F}} \widehat{\mathbb{X}}_p/\mathfrak{m}_f^{[p]} \geq r+1$ by (5.4). Hence one obtains

$$\dim_{\mathbb{F}} H^1(X^{[p]} \otimes \overline{\mathbb{Q}_{p^2}}, \mathscr{F})(1)/\mathfrak{m}_f^{[p]} \geq 2r+1.$$

This gives a contradiction. $\square$

Since $\lambda \nmid (p^2-1)$, we have the identifications

$$H^1(\mathbb{Q}_{p^2}, \mathbb{X}_p/\mathfrak{m}_f^{[p]}) = \mathrm{Hom}_{\mathrm{unr}}(\mathrm{Gal}(\overline{\mathbb{Q}_{p^2}}/\mathbb{Q}_{p^2}), \mathbb{X}_p/\mathfrak{m}_f^{[p]}) = \mathrm{Hom}(\mathcal{O}/\lambda\mathcal{O}, \mathbb{X}_p/\mathfrak{m}_f^{[p]}).$$

Therefore we have the exact sequence

$$\overline{\Phi}_p/\mathfrak{m}_f^{[p]} \to H^1(\mathbb{Q}_{p^2}, H^1(X^{[p]} \otimes \overline{\mathbb{Q}_{p^2}}, \mathscr{F})(1)/\mathfrak{m}_f^{[p]}) \to H^1_{\mathrm{unr}}(\mathbb{Q}_{p^2}, \mathbb{X}_p/\mathfrak{m}_f^{[p]}), \tag{5.5}$$

where $\overline{\Phi}_p/\mathfrak{m}_f^{[p]}$ is a quotient of $\Phi_p/\mathfrak{m}_f^{[p]}$. Recall that $H^1(\mathbb{Q}_{p^2}, H^1(X^{[p]} \otimes \overline{\mathbb{Q}_{p^2}}, \mathscr{F})(1)/\mathfrak{m}_f^{[p]})$ and it can be decomposed as the direct sum of two r-dimensional subspaces. Furthermore, one of the subspace is generated by unramified cohomology classes and the other by ramified cohomology classes. By Theorem 5.3, the group $\Phi_p/\mathfrak{m}_f^{[p]}$ is isomorphic to $\mathcal{O}/\lambda\mathcal{O}$. Hence by the exact sequence (5.5) we have $r=1$ and $\overline{\Phi}_p/\mathfrak{m}_f^{[p]} \cong \Phi_p/\mathfrak{m}_f^{[p]}$. Therefore $H^1(X^{[p]} \otimes \overline{\mathbb{Q}}, \mathscr{F})(1)/\mathfrak{m}_f^{[p]}$ is isomorphic to $T_{f,1}$.

Next we show that $H^1(X^{[p]} \otimes \overline{\mathbb{Q}}, \mathscr{F})(1)/\mathcal{I}_f^{[p]}$ is isomorphic to $T_{f,n}$. There is a natural $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant projection

$$H^1(X^{[p]} \otimes \overline{\mathbb{Q}}, \mathscr{F})(1)/\mathcal{I}_f^{[p]} \to H^1(X^{[p]} \otimes \overline{\mathbb{Q}}, \mathscr{F})(1)/\mathfrak{m}_f^{[p]}.$$

By the exact sequence

$$0 \to \widehat{\mathbb{X}}_p(1)/\mathcal{I}_f^{[p]} \to H^1(X^{[p]} \otimes \overline{\mathbb{Q}_{p^2}}, \mathscr{F})(1)/\mathcal{I}_f^{[p]} \to \mathbb{X}_p/\mathcal{I}_f^{[p]} \to 0$$

and the fact that the group $\Phi_p/\mathcal{I}_f^{[p]}$ is isomorphic to $\mathcal{O}_n$, we can take an element t_1 in $H^1(X^{[p]}\otimes\overline{\mathbb{Q}},\mathscr{F})(1)/\mathcal{I}_f^{[p]}$ which generates a subgroup C isomorphic to $\mathcal{O}_n$. Hence we can choose $t_1,t_2\in H^1(X^{[p]}\otimes\overline{\mathbb{Q}},\mathscr{F})(1)/\mathcal{I}_f^{[p]}$ such that $H^1(X^{[p]}\otimes\overline{\mathbb{Q}},\mathscr{F})(1)/\mathcal{I}_f^{[p]}\cong\mathcal{O}_n\cdot t_1\bigoplus\mathcal{O}_r\cdot t_2$ with $r\le n$. Since the residual Galois representation $\overline{\rho}_f$ is absolutely irreducible, one has

$$\overline{\rho}_f(\mathbb{F}[G_{\mathbb{Q}}])=\mathrm{End}_{\mathbb{F}}(T_{f,1})=\mathrm{End}_{\mathcal{O}}(H^1(X^{[p]}\otimes\overline{\mathbb{Q}},\mathscr{F})(1)/\mathfrak{m}_f^{[p]}).$$

Therefore there exist $h\in\overline{\rho}_f(\mathbb{F}[G_{\mathbb{Q}}])$ such that $ht_2=at_1+bt_2$ with $a\in\mathcal{O}^\times$, $b\in\mathcal{O}$. This implies $r=n$ and $H^1(X^{[p]}\otimes\overline{\mathbb{Q}},\mathscr{F})(1)/\mathcal{I}_f^{[p]}$ is isomorphic to $\mathcal{O}_n^2$. Hence, $H^1(X^{[p]}\otimes\overline{\mathbb{Q}},\mathscr{F})(1)/\mathcal{I}_f^{[p]}$ is isomorphic to $T_{f,n}$. □

Let $\mathcal{O}_{K,m}=\mathbb{Z}+m\mathcal{O}_K$ be the order of the imaginary quadratic field K of conductor m. Let K_m be the ring class field of K of conductor m. Write $\Phi_{p,m}$ for $\bigoplus_{\mathfrak{p}|p}\Phi_{\mathfrak{p}}$, where the sum is taken over the primes $\mathfrak{p}$ of K_m and $\Phi_{\mathfrak{p}}$ denotes the component group associated to the Shimura curve $X^{[p]}$ and the lisse sheaf $\mathscr{F}$ at $\mathfrak{p}$. Since the prime p is inert in K, it splits completely in K_m/K. Hence, the choice of a prime of K_m above p identifies $\Phi_{p,m}$ with $\Phi_p[\mathcal{G}_m]$. Therefore, we have an isomorphism

$$\Phi_{p,m}/\mathcal{I}_f^{[p]}\cong\mathcal{O}_n[\mathcal{G}_m].$$

For $X=X^{[p]}$ or $X_d^{[p]}$, let X_η be the generic fiber of $X_{\mathbb{Z}_p}\otimes\mathbb{Z}_p^{\mathrm{ur}}$ and X_s the special fiber. For a $\mathbb{Q}_p^{\mathrm{ur}}$-valued point x on X, denote

$$H_x^2(X_\eta,\mathscr{F})(1)^0:=\mathrm{Ker}[H_x^2(X_\eta,\mathscr{F})(1)\to H^2(X_\eta,\mathscr{F})(1)\to H^2(X_{\overline{\eta}}\mathscr{F})(1)].$$

Then we have a canonical map $H_x^2(X_\eta,\mathscr{F})(1)^0\to H^2(X_\eta,\mathscr{F})(1)^0:=\mathrm{Ker}[H^2(X_\eta,\mathscr{F})(1)\to H^2(X_{\overline{\eta}}\mathscr{F})(1)]$. Let $I_{\mathbb{Q}_p}$ be the inertia group and $I_{\mathbb{Q}_p}^t$ the tame inertia. By the Hochschild-Serre spectral sequence

$$E_2^{i,j}=H^i(I_{\mathbb{Q}_p},H^j(X_{\overline{\eta}},\mathscr{F})(1))\Rightarrow H^{i+j}(X_\eta,\mathscr{F})(1)$$

we obtain a map $H^2(X_\eta,\mathscr{F})(1)^0\to H^1(I_{\mathbb{Q}_p},H^1(X_{\overline{\eta}},\mathscr{F})(1))$. Assume that $d\ge 4$. Since $X_{d,\mathbb{Z}_p}^{[p]}$ is semistable, $R\Psi\mathscr{F}$ is tame (Illusie [17, Theorem 1.2]). Therefore this map induces

$$\alpha: H^2(X_{d,\eta}^{[p]},\mathscr{F})(1)^0\to H^1(I_{\mathbb{Q}_p}^t,H^1(X_{d,s}^{[p]},R\Psi\mathscr{F})(1))\cong H^1(I_{\mathbb{Q}_p}^t,H^1(X_{d,\overline{\eta}}^{[p]},\mathscr{F})(1)).$$

On the other hand, we have a map $H_x^2(X_{d,\eta}^{[p]},\mathscr{F})(1)^0\to H^1(I_{\mathbb{Q}_p}^t,H^1(X_{d,\overline{\eta}}^{[p]},\mathscr{F})(1))$ by the composition of maps

$$H_x^2(X_{d,\eta}^{[p]},\mathscr{F})(1)^0\longrightarrow H_x^2(X_{d,\overline{\eta}}^{[p]},\mathscr{F})(1)^0\xrightarrow{d_p}\Phi_{d,p}\xrightarrow{\beta}H^1(I_{\mathbb{Q}_p}^t,H^1(X_{d,\overline{\eta}}^{[p]},\mathscr{F})(1)),$$

where the map β is induced by the monodromy pairing

$$H^0(I_{\mathbb{Q}_p}^t,\mathbb{X}_{d,p})(\cong\mathbb{X}_{d,p})\xrightarrow{\lambda}H^1(I_{\mathbb{Q}_p}^t,H^1(X_{d,s}^{[p]},i^*\mathscr{F})(1))(\cong\widehat{\mathbb{X}}_{d,p})\to H^1(I_{\mathbb{Q}_p}^t,H^1(X_{d,\overline{\eta}}^{[p]},\mathscr{F})(1)).$$

Theorem 5.6. (1) *Let x be a $\mathbb{Q}_p^{\mathrm{ur}}$-valued point on $X^{[p]}$ such that x mod p is a non-singular point. Then, there is a commutative diagram*

$$\begin{tikzcd}
H_x^2(X_\eta^{[p]},\mathscr{F})(1)^0 \arrow[r] \arrow[d] & H^2(X_\eta^{[p]},\mathscr{F})(1)^0 \arrow[d, "\alpha"] \\
\Phi_p \arrow[r, "\beta"] & H^1(I_{\mathbb{Q}_p}^t,H^1(X_{\overline{\eta}}^{[p]},\mathscr{F})(1)).
\end{tikzcd}$$

(2) *The map β induces an isomorphism*

$$\Phi_p/\mathcal{I}_f^{[p]}\simeq H^1_{\mathrm{sing}}(\mathbb{Q}_{p^2},T_{f,n}).$$

Proof. For the first part, it is enough to show the commutativity of the following diagram:

$$\begin{tikzcd}
H^2(X_s^{[p]},i^*\mathscr{F})(1)^0 \arrow[r, "\mathrm{sp}(1)"] \arrow[d, "\omega_p"] & H^2(X_\eta^{[p]},\mathscr{F})(1)^0 \arrow[d, "\alpha"] \\
\Phi_p \arrow[r, "\beta"] & H^1(I_{\mathbb{Q}_p}^t,H^1(X_{\overline{\eta}}^{[p]},\mathscr{F})(1)).
\end{tikzcd}$$

Fix a topological generator σ of the tame inertia $I^t_{\mathbb{Q}_p}$. First we work with the Shimura curve $X_d^{[p]}$ instead of $X^{[p]}$. By [31, Lemma 1.6], we have a distinguished triangle

$$\to i^*Rj_*\Lambda \to R\Psi\Lambda \xrightarrow{\sigma-1} R\Psi\Lambda \xrightarrow{+1},$$

where $\Lambda = \mathbb{Z}_\ell$. Since the action of σ on $i^*\mathscr{F}$ is trivial and $\mathscr{F}$ is extended to the model of X smoothly, we have an isomorphism $i^*\mathscr{F}\otimes R\Psi\Lambda \cong R\Psi\mathscr{F}$. Therefore one has a distinguished triangle

$$\to i^*Rj_*\mathscr{F} \to R\Psi\mathscr{F} \xrightarrow{\sigma-1} R\Psi\mathscr{F} \xrightarrow{+1}.$$

Let γ be the composition of morphisms

$$R\Phi\mathscr{F} \to \bigoplus_{x\in\Sigma}(R\Phi\mathscr{F})_x \xrightarrow{\oplus_x \mathrm{Var}(\sigma)_x} \bigoplus_{x\in\Sigma} i_{x*}i_x^! R\Psi\mathscr{F} \xrightarrow{\mathrm{adj}} R\Psi\mathscr{F}.$$

Then we have the following commutative diagram:

$$\begin{tikzcd}
{} \arrow[r] & i^*\mathscr{F} \arrow[r] \arrow[d] & R\Psi\mathscr{F} \arrow[r] \arrow[d, equal] & R\Phi\mathscr{F} \arrow[r, "+1"] \arrow[d, "\gamma"] & {} \\
{} \arrow[r] & i^*Rj_*\mathscr{F} \arrow[r] & R\Psi\mathscr{F} \arrow[r, "\sigma-1"] & R\Psi\mathscr{F} \arrow[r, "+1"] & {}
\end{tikzcd}.$$

Taking cohomology $H^i(X_{d,s}^{[p]}, -)$, one has the following commutative diagram:

$$\begin{tikzcd}
H^1(X_{d,s}^{[p]}, R\Psi\mathscr{F}) \arrow[r] \arrow[d, equal] & H^1(X_{d,s}^{[p]}, R\Phi\mathscr{F}) \arrow[r] \arrow[d, "\gamma'"] & H^2(X_{d,s}^{[p]}, i^*\mathscr{F}) \arrow[r] \arrow[d, "\cong"] & H^2(X_{d,s}^{[p]}, R\Psi\mathscr{F}) \arrow[d, equal] \\
H^1(X_{d,s}^{[p]}, R\Psi\mathscr{F}) \arrow[r] \arrow[d, equal] & H^1(X_{d,s}^{[p]}, R\Psi\mathscr{F}) \arrow[r] \arrow[d, equal] & H^2(X_{d,s}^{[p]}, i^*Rj_*\mathscr{F}) \arrow[r] \arrow[d, equal] & H^2(X_{d,s}^{[p]}, R\Psi\mathscr{F}) \arrow[d, equal] \\
H^1(X_{d,\overline{\eta}}^{[p]}, \mathscr{F}) \arrow[r, "\sigma-1"] & H^1(X_{d,\overline{\eta}}^{[p]}, \mathscr{F}) \arrow[r, "\delta"] & H^2(X_{d,\eta}^{[p]}, \mathscr{F}) \arrow[r] & H^2(X_{d,\overline{\eta}}^{[p]}, \mathscr{F})
\end{tikzcd},$$

where γ' is the composition of morphisms

$$H^1(X_{d,s}^{[p]}, R\Phi\mathscr{F}) = \bigoplus_{x\in\Sigma}(R\Phi\mathscr{F})_x \xrightarrow{\mathrm{Var}(\sigma)} \bigoplus_{x\in\Sigma} H^1_x(X_{d,s}^{[p]}, R\Psi\mathscr{F}) \to H^1(X_{d,s}^{[p]}, R\Psi\mathscr{F}).$$

Then one can see that $\delta : H^1(X_{d,\overline{\eta}}^{[p]}, \mathscr{F}) \to H^2(X_{d,\eta}^{[p]}, \mathscr{F})$ factors through the coinvariant $H^1(X_{d,\overline{\eta}}^{[p]}, \mathscr{F})_{\sigma-1} \cong H^1(I^t_{\mathbb{Q}_p}, H^1(X_{d,\overline{\eta}}^{[p]}, \mathscr{F}))$ and the map $H^1(I^t_{\mathbb{Q}_p}, H^1(X_{d,\overline{\eta}}^{[p]}, \mathscr{F})) \to H^2(X_{d,\eta}^{[p]}, \mathscr{F})$ coincides with the inverse of the map obtained by Hochschild-Serre spectral sequence. Applying the projector ϵ_d defined by

$$\epsilon_d := \frac{1}{\#G_d}\sum_{g\in G_d} g \in \mathbb{Q}[G_d],$$

the first part of the theorem follows. Since $T_{f,n}$ is unramified at p and λ does not divide p, one has $H^1(I_{\mathbb{Q}_p}, H^1(X_{\overline{\eta}}^{[p]}, \mathscr{F})(1)/\mathcal{I}_f^{[p]}) \cong H^1(I^t_{\mathbb{Q}_p}, H^1(X_{\overline{\eta}}^{[p]}, \mathscr{F})(1)/\mathcal{I}_f^{[p]})$. Therefore the second part follows from the discussions in the proof of Proposition 5.4. □

6. Kuga-Sato varieties and CM-cycles

6.1. **The ℓ-adic Abel-Jacobi map.** Here, we recall some basic facts on ℓ-adic Abel-Jacobi map following Jannsen ([18], [19]).

Let Y be a proper smooth variety over a field F of characteristic zero. For an integer $i \geq 0$, write $CH^i(Y/F)$ for the Chow group of algebraic cycles defined over F of codimension i on Y modulo rational equivalence. Fix a rational prime ℓ. Then one may define the cycle class map

$$\mathrm{cl}_\ell : CH^i(Y/F) \to H^{2i}(Y_{\overline{F}}, \mathbb{Z}_\ell(i))^{G_F}$$

and we denote by $CH^i(Y)_0$ its kernel. Note that this definition does not depend on the choice of the prime ℓ by Lefschetz principle and comparison theorem between étale cohomology and singular cohomology.

The cycle class map cl_ℓ factors through $H^{2i}(Y,\mathbb{Z}_\ell(i))$, then the Hochschild-Serre spectral sequence

$$H^i(F,H^j(Y_{\overline{F}},\mathbb{Z}_\ell(k)))\Rightarrow H^{i+j}(Y,\mathbb{Z}_\ell(k))$$

induces the ℓ-adic Abel-Jacobi map

$$AJ_\ell : CH^i(Y/F)_0\to H^1(F,H^{2i-1}(Y_{\overline{F}},\mathbb{Z}_\ell(k))).$$

By Jannsen [18] we have the following geometric description of the ℓ-adic Abel-Jacobi map. Let Z be a homologically trivial cycle on X defined over F of codimension i representing an element in $CH^i(Y/F)_0$. The pull-back of the extension of G_F-modules

$$\begin{aligned}0\to H^{2i-1}(Y_{\overline{F}},\mathbb{Z}_\ell(i))\to H^{2i-1}(Y_{\overline{F}}\setminus|Z_{\overline{F}}|,\mathbb{Z}_\ell(i))&\\ \to \mathrm{Ker}\left[H^{2i}_{|Z_{\overline{F}}|}(Y_{\overline{F}},\mathbb{Z}_\ell(i))\to H^{2i}(Y_{\overline{F}},\mathbb{Z}_\ell(i))\right]&\to 0\end{aligned}$$

by the map $\mathbb{Z}_\ell\to H^{2i}_{|Z_{\overline{F}}|}(Y_{\overline{F}},\mathbb{Z}_\ell(i))$ sending 1 to $b(Z)$, where $b(Z)$ is the cohomology class of $Z_{\overline{F}}$.

6.2. **Kuga-Sato varieties over Shimura curves.** To construct global cohomology classes in $H^1(K_m,T_{f,n})$, we will use the image of algebraic cycles on Kuga-Sato varieties under the ℓ-adic Abel-Jacobi map. We keep the assumptions and notations as in §4. Now we suppose that d is a prime greater than 3 which splits in K and is prime to $N\ell p$. Let $\pi:\mathscr{A}_d^{[p]}\to X_d^{[p]}$ be the universal abelian surface over the Shimura curve $X_d^{[p]}$. Then we define the Kuga-Sato variety

$$\pi_k : W_{k,d}^{[p]}\to X_d^{[p]}$$

by the $\frac{k-2}{2}$-fold fiber product over $X_d^{[p]}$ of $\mathscr{A}_d^{[p]}$ with itself.

Since the action of $\mathcal{O}_{B'}$ on $\mathscr{A}_d^{[p]}$ induces an action of $B'^{\times}$ on $R^i\pi_*\mathbb{Q}_\ell$, one may define

$$\mathcal{L}_2 := \bigcap_{b\in B'}\mathrm{Ker}\left[b-1 : R^2\pi_*\mathbb{Q}_\ell\to R^2\pi_*\mathbb{Q}_\ell\right]$$

following Iovita-Spiess [16]. For an integer $m\geq 2$, let

$$\Delta_m : \mathrm{Sym}^m\,\mathcal{L}_2\to \mathrm{Sym}^{m-2}\,\mathcal{L}_2(-2)$$

be the Laplace operator symbolically given by

$$\Delta_m(x_1\cdots x_m)=\sum_{1\leq i<j\leq m}(x_i,x_j)x_1\cdots\widehat{x}_i\cdots\widehat{x}_j\cdots x_m,$$

where $(\ ,\ )$ is the non-degenerate pairing

$$(\ ,\ ) : \mathcal{L}_2\times\mathcal{L}_2\hookrightarrow R^2\pi_*\mathbb{Q}_\ell\otimes R^2\pi_*\mathbb{Q}_\ell\overset{\cup}{\to}R^4\pi_*\mathbb{Q}_\ell\overset{\mathrm{Tr}}{\to}\ \mathbb{Q}_\ell(-2).$$

Let $\mathcal{L}_{k-2}$ denote the kernel of $\Delta_{\frac{k-2}{2}}$.

Then there exists a projector ϵ_k defined as in Scholl [32] (also see Iovita-Spiess[16, §10]) such that

$$\epsilon_d\cdot\epsilon_kH^{k-1}(W_{k,d}^{[p]}\otimes\overline{\mathbb{Q}},\mathbb{Q}_\ell)\otimes_{\mathbb{Q}_\ell}E\cong\epsilon_dH^1(X_d^{[p]}\otimes\overline{\mathbb{Q}},\mathcal{L}_{k-2})\otimes_{\mathbb{Q}_\ell}E\cong H^1(X^{[p]}\otimes\overline{\mathbb{Q}},\mathscr{F})(1)\otimes_{\mathcal{O}}E,$$

where ϵ_d is the projector defined by

$$\epsilon_d=\frac{1}{\#G_d}\sum_{g\in G_d}g\in\mathbb{Q}[G_d].$$

Note that

$$\epsilon_kH^{k-1}(W_{k,d}^{[p]}\otimes\overline{\mathbb{Q}},\mathbb{Q}_\ell)\simeq\epsilon_kH^*(W_{k,d}^{[p]}\otimes\overline{\mathbb{Q}},\mathbb{Q}_\ell).$$

6.3. **Description on CM points.** By the moduli interpretation of the Shimura curve $X^{[p]}$, a point on $X^{[p]}$ is represented by a triple (A, ι, C). For $m \geq 0$, there exists a point $P_m = (A_m, \iota_m, C_m)$ such that $\mathrm{End}(P_m)$ is isomorphic to $\mathcal{O}_{K,m}$, where $\mathrm{End}(P_m)$ is the ring of endomorphisms of A_m which commutes with the action of ι_m and respect the level structure C_m and $\mathcal{O}_{K,m}$ is the order of K of conductor m. The point P_m is called a CM point of level m. By the theory of complex multiplication, such point P_m is defined over K_m, where K_m is the ring class field of K of conductor m.

Using the complex uniformization of the Shimura curve $X^{[p]}$, the CM points of level m are defined by

$$P_m(a) := \left[(z', \varphi_{B,B'}(a^{(p)}\varsigma\tau^{N^+}))\right]_{\mathbb{C}} \in X^{[p]}(\mathbb{C})$$

for each $a \in \widehat{K}^\times$. By Shimura's reciprocity law, one has

$$P_m(a) \in \mathrm{CM}_K^{p-\mathrm{ur}}(X^{[p]}) \cap X^{[p]}(K_m)$$

and $P_m(a)^\sigma = P_m(ab)$ for $\sigma = \mathrm{rec}_K(b) \in \mathcal{G}_m$. Set $P_m = P_m(1)$.

6.4. **Definition of CM cycles.** Here we construct CM cycles following Nekovář [24] and Iovita-Spiess [16]. Let $X = X_{N^+,N^-}$ be the Shimura curve defined in §4 and let $P_m = (A_m, \iota_m, C_m)$ be a CM point of level m. Then A_m is defined over the ring class field K_m. Write $NS(A_m)$ for the Néron-Severi group of A_m. There is a natural right $B'^\times$-action on $NS(A_m)_{\mathbb{Q}}$ given by $\mathcal{L} \cdot b = \mathrm{Nrd}(b)^{-1}\iota_m(b)^*(\mathcal{L})$ for $b \in B'^\times$ and $\mathcal{L} \in NS(A_m)_{\mathbb{Q}}$. Note that our normalization is different from the action used in [16].

Since $\mathrm{End}(P_m) \simeq \mathcal{O}_{K,m}$ and A_m has endomorphism by the maximal order $\mathcal{O}_{B'}$, A_m has endomorphisms by an order $\mathcal{O}_{B'} \otimes \mathcal{O}_{K,m}$ in $B' \otimes K \simeq M_2(K)$. Hence A_m is isogenous to a product $E_m \times E_m$, where E_m is an elliptic curve with complex multiplication by $\mathcal{O}_{K,m}$. Write Γ_m for the graph of $m\sqrt{D_K}$. Then, define Z_m to be the image of the divisor $[\Gamma_m] - [E_m \times 0] - m^2|D_K|[0 \times E_m]$ in $NS(A_m)$. It lies in the free rank one $\mathbb{Z}$-module $\langle [E_m \times 0], [0 \times E_m], \Delta_{E_m} \rangle^\perp \subset NS(A_m)$, where Δ_{E_m} is the diagonal.

Proposition 6.1. *Assume that A has complex multiplication by $\mathcal{O}_{K,m}$. Then there exists an element y_m in $NS(A) \otimes \mathbb{Q}$ such that*

(1) *$\iota_m(b)^*(y_m) = y_m$ for any $b \in B'^\times$,*
(2) *The self-intersection number of y_m is $2D_K$.*

Moreover, y_m is uniquely determined up to sign by these properties.

Proof. This is a direct generalization of [16, Proposition 8.2]. In particular, $y_m = m^{-1}Z_m$ satisfies the properties. ☐

Remark 6.2. Since we use a different normalization for the action of $B'^\times$ on $NS(A_m)_{\mathbb{Q}}$ with [16], the formula (1) in Proposition 6.1 is different from the corresponding formula in [16, Proposition 8.2].

Let t denote the number of prime divisors of Np, h_m the class number of K_m. Then there are exactly $2^t h_m$ CM points of conductor m (see Bertolini-Darmon [2] for details).

Let $\mathcal{W}$ be the Atkin-Lehner group of order 2^t generated by all the Atkin-Lehner involutions W_q^+ with $q \mid N^+$ and W_q^- with $q \mid N^-p$. Write $\mathcal{G}_m$ for the Galois group $\mathrm{Gal}(K_m/K)$. One can identify the Galois group $\mathcal{G}_m$ with $\mathrm{Pic}(\mathcal{O}_{K,m})$ via geometrically normalized reciprocity map. The group $\mathrm{Pic}(\mathcal{O}_{K,m}) \times \mathcal{W}$ acts simply transitively on the set of CM points.

Recall that $d \geq 4$ is an integer relatively prime to Np and

$$\pi : \mathscr{A} = \mathscr{A}_d^{[p]} \to X_d^{[p]}$$

is the universal abelian surface and

$$\psi : X_d^{[p]} \to X^{[p]}$$

the natural morphism. Let $P_m = P_m(1)$ be the CM point of level m defined as above and let $\widetilde{P}_m$ be any point on $X_d^{[p]}$ such that $\psi(\widetilde{P}_m) = P_m$. The fiber $A_{\widetilde{P}_m} = \pi^{-1}(\widetilde{P}_m)$ is an abelian surface with $\mathrm{End}_{\mathcal{O}_{B'}}(A_{\widetilde{P}_m}) \simeq \mathcal{O}_{K,m}$. By Proposition 6.1, there exist an element $y_m \in NS(A_{\widetilde{P}_m})_{\mathbb{Q}}$ satisfying

(1) $\iota_m(b)^*(y_m) = y_m$ for any $b \in B'^\times$,
(2) The self-intersection number of y_m is $2D_K$.

which is uniquely determined up to sign.

Let $Y_{\widetilde{P}_m}$ be an element of $\epsilon_4 CH^1(A_{\widetilde{P}_m})_{\mathbb{Q}}$ representing $y_{\widetilde{P}_m}$. One may choose the elements $Y_{\widetilde{P}_m}$ satisfying that

$$g_*(Y_{\widetilde{P}_m}) = Y_{g_*(\widetilde{P}_m)} \text{ for all } \widetilde{P}_m \in \psi^{-1}(P_m) \text{ and } g \in G_d,$$

where $g : A_{\widetilde{P}_m} \to A_{\widetilde{P}_m}$ is the automorphism induced by $g \in G_d$.

Let $j_{k,m} : A_{\widetilde{P}_m}^{\frac{k-2}{2}} \hookrightarrow \mathscr{A}^{\frac{k-2}{2}} = W_{k,d}^{[p]}$ be the inclusion of the fiber over $\widetilde{P}_m$ into the Kuga-Sato variety. We define the element $Z_{\widetilde{P}_m}$ of $\epsilon_d \cdot \epsilon_4 CH^2(\mathscr{A}/K_m)_{\mathbb{Q}}$ by the image of $Y_{\widetilde{P}_m}$ under the composition of maps

$$\epsilon_4 CH^1(A_{\widetilde{P}_m})_{\mathbb{Q}} \xrightarrow{j_{4,m}} \epsilon_4 CH^2(\mathscr{A}/K_m)_{\mathbb{Q}} \xrightarrow{\epsilon_d} \epsilon_d \cdot \epsilon_4 CH^2(\mathscr{A}/K_m)_{\mathbb{Q}}.$$

We require that the elements $Y_{\widetilde{P}_m}$ satisfy the compatibility with the action of $\mathcal{W} \times \mathrm{Pic}(\mathcal{O}_{K,m})$ (see Iovita-Spiess [16, page 366] for details). Then we define the CM cycle $Z_m^{\frac{k-2}{2}}$ of level m by setting

$$Z_m^{\frac{k-2}{2}} := \epsilon_d \cdot \epsilon_k (j_{k,m})_*(Y_{\widetilde{P}_m}^{\frac{k-2}{2}}) \in \epsilon_d \cdot \epsilon_k CH^{k/2}(W_{k,d}^{[p]}/K_m)_{\mathbb{Q}} \subset CH^{k/2}(W_{k,d}^{[p]}/K_m)_{\mathbb{Q}}.$$

7. Construction of Euler systems and the explicit reciprocity law

7.1. **Construction of special cohomology classes.** Let p be an n-admissible prime. Here we give a description of the image of CM cycles under the ℓ-adic Abel-Jacobi map following Nekovář [25]. Write Z_m for the CM cycle of level m. Let $\kappa_d^{[p]}(m)$ be the image of CM cycle $Z_m^{\frac{k-2}{2}}$ under the ℓ-adic Abel-Jacobi map

$$\epsilon_k \circ AJ_{\ell,E} : \mathrm{CH}^{k/2}(W_{k,d}^{[p]}/K_m)_E \to H^1(K_m, H^1(X_d^{[p]} \otimes \overline{K_m}, \mathscr{F})(1))_E.$$

By the construction of the cohomology class, we have the following lemma.

Lemma 7.1. *The global cohomology class* $\kappa^{[p]}(m) := \epsilon_d \kappa_d^{[p]}(m)$ *belongs to* $H^1(K_m, H^1(X^{[p]} \otimes \overline{K_m}, \mathscr{F})(1))$.

Let $\widetilde{P}_m$ be a lift of the CM point P_m of level m. Let

$$\mathrm{cl}_\ell : CH^{\frac{k-2}{2}}(A_{\widetilde{P}_m}^{\frac{k-2}{2}}) \to \epsilon_k H^{k-2}(A_{\widetilde{P}_m}^{\frac{k-2}{2}} \otimes \overline{K_m}, \mathbb{Z}_\ell(\frac{k-2}{2}))^{G_{K_m}}$$

be the cycle class map. Then $\epsilon_d \cdot \epsilon_k H^{k-2}(A_{\widetilde{P}_m}^{\frac{k-2}{2}} \otimes \overline{K_m}, \mathbb{Z}_\ell(\frac{k-2}{2}))_{\mathcal{O}}^{G_{K_m}}$ is isomorphic to $H^2_{P_m}(X^{[p]} \otimes \overline{K_m}, \mathscr{F})(1)^{G_{K_m}}$. By the similar argument of [25, Proof of (2.4) Proposition (2)], one can show that the image of $Y = Y_{\widetilde{P}_m}^{\frac{k-2}{2}}$ is represented by the pull-back of extension

$$\begin{aligned} 0 \to H^1(X_d^{[p]} \otimes \overline{K_m}, \mathscr{F})(1) &\to H^1(X_d^{[p]} \otimes \overline{K_m} \setminus \widetilde{P}_m \otimes \overline{K_m}, \mathscr{F})(1) \\ &\to H^2_{\widetilde{P}_m \otimes \overline{K_m}}(X_d^{[p]} \otimes \overline{K_m}, \mathscr{F})(1)^{G_{K_m}} \to 0 \end{aligned}$$

by the map $\mathcal{O} \to H^2_{\widetilde{P}_m \otimes \overline{K_m}}(X_d^{[p]} \otimes \overline{K_m}, \mathscr{F})(1)^{G_{K_m}}$ sending 1 to $\epsilon_k b(Y)$, where $b(Y)$ is the cohomology class of $Y_{\overline{K_m}}$. We will compute the image

$$\epsilon_d \cdot \epsilon_k b(Y) \in H^2_{P_m \otimes \overline{K_m}}(X^{[p]} \otimes \overline{K_m}, \mathscr{F})(1).$$

Recall that there is an elliptic curve E_m with complex multiplication by $\mathcal{O}_{K,m}$ defined over K_m such that $A_{\widetilde{P}_m}$ is isogenous to $E_m \times E_m$. Then, Künneth formula and antisymmetrization gives a projection

$$\begin{aligned} \mathrm{pr}_k : H^{k-2}(A_{\widetilde{P}_m}^{\frac{k-2}{2}} \otimes \overline{K_m}, \mathbb{Z}_\ell(k/2-1)) &\to H^1(E_m \otimes \overline{K_m}, \mathbb{Z}_\ell)^{\otimes k-2}(k/2-1) \\ &\to (\mathrm{Sym}^{k-2} H^1(E_m \otimes \overline{K_m}, \mathbb{Z}_\ell))(k/2-1). \end{aligned}$$

One obtains that the element $\epsilon_k \mathrm{cl}_\ell(Y)$ belongs to the space $(\mathrm{Sym}^{k-2} H^1(E_m \otimes \overline{K_m}, \mathbb{Q}_\ell))(k/2-1)$. There exists a $B_\ell^\times \simeq B'^\times_\ell \simeq \mathrm{GL}_2(\mathbb{Q}_\ell)$-equivariant isomorphism

$$\begin{aligned}(\mathrm{Sym}^{k-2} H^1(E_m \otimes \overline{K_m}, \mathbb{Q}_\ell))(k/2-1) &\xrightarrow{\simeq} (\mathrm{Sym}^{k-2} H^1(E_m \otimes \overline{K_{m,\mathfrak{p}}}, \mathbb{Q}_\ell))(k/2-1)\\ &\xrightarrow{\simeq} (\mathrm{Sym}^{k-2} H^1(E_m \otimes \overline{\mathbb{F}_{p^2}}, \mathbb{Q}_\ell))(k/2-1)\\ &\xrightarrow{\simeq} L_k(\mathbb{Q}_\ell)\end{aligned}$$

which preserves the intersection pairing. Therefore, we have an identification

$$H^2_{\widetilde{P}_m \otimes \overline{\mathbb{Q}_{p^2}}}(X_d^{[p]} \otimes \overline{\mathbb{Q}_{p^2}}, \mathscr{F})(1) \cong H^2_{\overline{\widetilde{P}}_m \otimes \overline{\mathbb{F}_{p^2}}}(X_d^{[p]} \otimes \overline{\mathbb{F}_{p^2}}, \mathscr{F})(1) \cong L_k(\mathcal{O}),$$

where $\overline{\widetilde{P}}_m = \widetilde{P}_m \mod p$.

Lemma 7.2. *The image of $\epsilon_d \cdot \epsilon_k \mathrm{cl}_\ell(Y)$ in $L_k(\mathbb{Q}_\ell)$ is given by $\boldsymbol{v}_0^*$ up to sign.*

Proof. This follows from the fact that both elements satisfy the same properties:

1. (1) $\epsilon_d \cdot \epsilon_k \mathrm{cl}_\ell(Y)$ and $\boldsymbol{v}_0^*$ are eigenvectors for the action of K with eigenvalue 1.
2. (2) $\langle \epsilon_d \cdot \epsilon_k \mathrm{cl}_\ell(Y), \epsilon_d \cdot \epsilon_k \mathrm{cl}_\ell(Y)\rangle = \langle \boldsymbol{v}_0^*, \boldsymbol{v}_0^*\rangle = D_K^{k-2}$.

These properties characterize an element in $L_k(\mathbb{Q}_\ell)$ up to sign. □

By Theorem 5.3, we have an isomorphism $H^1(X^{[p]} \otimes \overline{K_m}, \mathscr{F}(1))/\mathcal{I}_f^{[p]} \simeq T_{f,n}$ as $\mathrm{Gal}(\overline{K_m}/K_m)$-modules, therefore $\sum_{\sigma \in \mathcal{G}_m} \epsilon_d \kappa^{[p]}(m)^\sigma$ defines a cohomology class $\kappa_{f,n}^{[p]}(m)$ in $H^1(K_m, T_{f,n})$.

7.2. **The explicit reciprocity law.** By Theorem 5.6 and the description of the ℓ-adic Abel-Jacobi map considered in the previous section, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^{k/2}(W_{k,d}^{[p]}/K_m) & \xrightarrow{\epsilon_d \cdot \epsilon_k AJ_\ell} & H^1(K_m, H^1(X^{[p]} \otimes \overline{K_m}, \mathscr{F})(1)) \\ \Big\downarrow{\scriptstyle \epsilon_d \cdot \epsilon_k \mathrm{cl}_\ell} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\ \bigoplus_{\mathfrak{p}|p} H^2(X^{[p]} \otimes K_{m,\mathfrak{p}}, \mathscr{F})(1) & \xrightarrow{\omega_p} & H^1_{\mathrm{sing}}(K_{m,p}, T_{f,n}). \end{array}$$

Proposition 7.3. *There exists a positive integer M such that*

$$\mathrm{red}_{\lambda^n}(\kappa_{f,n+M}^{[p]}(m)) \in H^1_p(K_m, T_{f,n}),$$

where p is a $n+M$-admissible prime.

Proof. For $v|dN^+$, by Lemma 2.3 and [4, Corollary 5.2] we have

$$\mathrm{red}_{\lambda^n}(\mathrm{res}_v \kappa_{f,n+M}^{[p]}(m)) \in H^1_f(K_{m,v}, T_{f,n})$$

for sufficiently large M. For $v|N^-$, since $H^0(K_{m,v}, A_f)$ is finite, $H^2(K_{m,v}, T_f) = \mathrm{Hom}(H^0(K_{m,v}, A_f), E/\mathcal{O})$ is also finite. Hence for sufficiently large M, $\lambda^M H^2(K_{m,v}, T_f) = 0$. The commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_f & \xrightarrow{\times \lambda^{n+M}} & T_f & \xrightarrow{\mathrm{red}_{\lambda^{n+M}}} & T_{f,n} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \times\lambda^M} & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle \mathrm{red}_{\lambda^n}} & & \\ 0 & \longrightarrow & T_f & \xrightarrow{\times \lambda^{n}} & T_f & \xrightarrow{\mathrm{red}_{\lambda^{n}}} & T_{f,n} & \longrightarrow & 0 \end{array}$$

gives rise to

$$\begin{array}{ccccc} H^1(K_{m,v}, T_f) & \xrightarrow{\mathrm{red}_{\lambda^{n+M}}} & H^1(K_{m,v}, T_{f,n}) & \longrightarrow & H^2(K_{m,v}, T_f)[\lambda^{n+M}] \\ \Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle \mathrm{red}_{\lambda^n}} & & \Big\downarrow{\scriptstyle \times\lambda^M} \\ H^1(K_{m,v}, T_f) & \xrightarrow{\mathrm{red}_{\lambda^{n}}} & H^1(K_{m,v}, T_{f,n}) & \longrightarrow & H^2(K_{m,v}, T_f)[\lambda^n]. \end{array}$$

Therefore by the definition of $H^1_f(K_{m,v}, T_{f,n})$ and the fact $H^1_f(K_{m,v}, V_f) = H^1(K_{m,v}, V_f)$ (see Besser [4, Proposition 4.1 (2)]), we have

$$\mathrm{red}_{\lambda^n}(\mathrm{res}_v(\kappa^{[p]}_{f,n+M}(m))) \in H^1_f(K_{m,v}, T_{f,n}).$$

For $v \nmid Nd\ell p$, the CM cycle $Z^{\frac{k-2}{2}}$ is unramified at v, hence the class $\mathrm{red}_{\lambda^n}(\kappa^{[p]}_{f,n+M}(m))$ is also unramified at v. For the case $v|\ell$, the Galois representation $H^1(X^{[p]}_{K_m}, \mathscr{F}(1))$ is crystalline, since the Kuga-Sato variety $W^{[p]}_{k,d}$ has good reduction at v. Hence by Nekovář [26, Theorem 3.1(1)] and Nizioł [29, Theorem 3.2], the image of the ℓ-adic Abel-Jacobi map is contained in H^1_f (also see Nekovář [26, Theorem 3.1(2)] and Nekovář-Nizioł [28, Theorem B] for general case). Therefore one has

$$\mathrm{red}_{\lambda^n}(\mathrm{res}_v(\kappa^{[p]}_{f,n+M}(m))) \in H^1_f(K_{m,v}, H^1(X^{[p]} \otimes \overline{K_{m,v}}, \mathscr{F})(1)/\mathcal{I}^{[p]}_f).$$

Since the prime ℓ is greater than $k-1$, one can use the Fontaine-Laffaille theory. Choose a Galois stable lattice T in a crystalline representation of $G_{K_{m,v}}$ such that $T/\lambda^n T \cong H^1(X^{[p]} \otimes K_{m,v}, \mathscr{F})(1)/\mathcal{I}^{[p]}_f$. Denote $T_1 = T$ and $T_2 = T_f$. Let D_i be a strongly divisible $\mathcal{O}$-lattice in $D_{\mathrm{cris}}(V_i) = D_{\mathrm{dR}}(V_i)$ (the equality follows from the facts that $K_{m,v} = \mathbb{Q}_{p^2}$ is an unramified extension of $\mathbb{Q}_p$ and V_i are crystalline) for $i = 1, 2$, where $V_i = T_i \otimes_{\mathcal{O}} E$. Define $D_i^{k/2} = D_i \cap D^{k/2}_{\mathrm{dR}}(V_i)$ and $\phi_{k/2} = \lambda^{-k/2}\phi$, where ϕ is the Frobenius morphism. By the Fontaine-Laffaille theory, we have isomorphisms $D_1/\lambda^n D_1 \cong D_2/\lambda^n D_2$ and $D_1^{k/2}/\lambda^n D_1^{k/2} \cong D_2^{k/2}/\lambda^n D_2^{k/2}$. Moreover by Bloch-Kato [5, Lemma 4.5 (c)], $h^1(D_i) = \mathrm{Coker}[D_i^{k/2} \overset{1-\phi_{k/2}}{\to} D_i]$ is isomorphic to $H^1_f(K_{m,v}, T_i)$. From these facts, it is easy to see

$$H^1_f(K_{m,v}, H^1(X^{[p]} \otimes K_{m,v}, \mathscr{F})(1)/\mathcal{I}^{[p]}_f) \cong H^1_f(K_{m,v}, T_{f,n})$$

for $v|\ell$. Therefore we have $\mathrm{red}_{\lambda^n}(\mathrm{res}_v(\kappa^{[p]}_{f,n+M}(m))) \in H^1_f(K_{m,v}, T_{f,n})$ for $v|\ell$. This completes the proof. □

The relation between the image of the CM cycle in $H^1_{\mathrm{sing}}(K_{m,p}, T_{f,n})$ and the theta element $\Theta(f_{\pi'})$ is given by the following theorem.

Theorem 7.4. *There exists a constant $u \in \mathcal{O}_n^\times$ such that $\partial_p(\mathrm{red}_{\lambda^n}(\kappa^{[p]}_{n+M}(m))) \equiv u \cdot \Theta(f_{\pi'}) \mod \lambda^n$.*

Proof. By Theorem 5.3, Theorem 5.6 and Lemma 7.2, one has

$$\partial_p(\kappa^{[p]}_{f,n+M}(m)) = \sum_{[a] \in \mathcal{G}_m} \langle v_0^*, \widehat{f}(x_m(a)\tau^{N^+}) \rangle_k \cdot [a]_m = \Theta(f_{\pi'}) \in \mathcal{O}_{n+M}[\mathcal{G}_m]$$

up to $\mathcal{O}^\times_{n+M}$. Therefore the natural image in $\mathcal{O}_n[\mathcal{G}_m]$ satisfies the same property. □

Now, our main result (Theorem 1.1) follows from Theorem 2.8 and Theorem 7.4

Remark 7.5. Assume that $\overline{\rho}_f$ is ramified at all primes dividing N^-. Then we have

$$\Omega_{\pi,N^-} = u \cdot \Omega_f^{\mathrm{can}} \text{ for some } u \in \mathcal{O}^\times.$$

This fact follows from Proposition 5.1 and the argument in [8, Proof of Proposition 6.1].

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