The Values of $\cos\left(\frac{2\pi}{9}\right)$, $\cos\left(\frac{4\pi}{9}\right)$, $\cos\left(\frac{6\pi}{9}\right)$, and $\cos\left(\frac{8\pi}{9}\right)$

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Let

$$\theta = \frac{2\pi}{9},$$

then

$$x_k = e^{ik\theta} \qquad k = 1,\ 2,\ \cdots,\ 8$$

are the solutions of

$$x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0. \tag{1}$$

Since

$$e^{5i\theta} = e^{-4i\theta}, \quad e^{6i\theta} = e^{-3i\theta}. \quad e^{7i\theta} = e^{-2i\theta}, \quad e^{8i\theta} = e^{-i\theta}.$$

equation (1) can be factorised as

$$\begin{aligned}
&x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 && (2)\\
&= \left(x - e^{i\theta}\right)\left(x - e^{-i\theta}\right)\left(x - e^{2i\theta}\right)\left(x - e^{-2i\theta}\right)\left(x - e^{3i\theta}\right)\left(x - e^{-3i\theta}\right)\left(x - e^{4i\theta}\right)\left(x - e^{-4i\theta}\right)\\
&= \left(x^2 - 2x\cos\theta + 1\right)\left(x^2 - 2x\cos 2\theta + 1\right)\left(x^2 - 2x\cos 3\theta + 1\right)\left(x^2 - 2x\cos 4\theta + 1\right) && (3)
\end{aligned}$$

For simplicity, we use the following shorthand notations

$$c_1 = \cos\theta, \quad c_2 = \cos 2\theta, \quad c_3 = \cos 3\theta, \quad c_4 = \cos 4\theta.$$

Then, (3) becomes

$$\begin{aligned}
&\left(x^2 - 2c_1x + 1\right)\left(x^2 - 2c_2x + 1\right)\left(x^2 - 2c_3x + 1\right)\left(x^2 - 2c_4x + 1\right) && (4)\\
&= x^8 - 2(c_1 + c_2 + c_3 + c_4)x^7\\
&\quad +4(1 + c_1c_2 + c_1c_3 + c_1c_4 + c_2c_3 + c_2c_4 + c_3c_4)x^6\\
&\quad -\Big(6(c_1 + c_2 + c_3 + c_4) + 8(c_1c_2c_3 + c_1c_2c_4 + c_1c_3c_4 + c_2c_3c_4)\Big)x^5\\
&\quad +\Big(6 + 8\left(c_1c_2 + c_1c_3 + c_1c_4 + c_2c_3 + c_2c_4 + c_3c_4\right) + 16c_1c_2c_3c_4\Big)x^4\\
&\quad -\Big(6(c_1 + c_2 + c_3 + c_4) + 8(c_1c_2c_3 + c_1c_2c_4 + c_1c_3c_4 + c_2c_3c_4)\Big)x^3\\
&\quad +4(1 + c_1c_2 + c_1c_3 + c_1c_4 + c_2c_3 + c_2c_4 + c_3c_4)x^2\\
&\quad -2(c_1 + c_2 + c_3 + c_4)x + 1 && (5)
\end{aligned}$$

Compairing the coefficients of (2) with those of (5), we find

$$c_1 + c_2 + c_3 + c_4 = -\frac{1}{2} \tag{6}$$

$$c_1c_2 + c_1c_3 + c_1c_4 + c_2c_3 + c_2c_4 + c_3c_4 = -\frac{3}{4} \tag{7}$$

$$c_1c_2c_3 + c_1c_2c_4 + c_1c_3c_4 + c_2c_3c_4 = \frac{1}{4} \tag{8}$$

$$c_1c_2c_3c_4 = \frac{1}{16} \tag{9}$$

Next, consider the equation

$$
\begin{aligned}
&(y-c_1)(y-c_2)(y-c_3)(y-c_4) \\
&= y^4-(c_1+c_2+c_3+c_4)y^3+(c_1c_2+c_1c_3+c_1c_4+c_2c_3+c_2c_4+c_3c_4)y^2 \\
&\quad -(c_1c_2c_3+c_1c_2c_4+c_1c_3c_4+c_2c_3c_4)y+c_1c_2c_3c_4 \\
&= y^4+\frac{1}{2}y^3-\frac{3}{4}y^2-\frac{1}{4}y+\frac{1}{16}
\end{aligned}
$$

Hence $\cos\theta$, $\cos 2\theta$, $\cos 3\theta$, and $\cos 4\theta$ can be obtained by solving the quartic equation,

$$16y^4+8y^3-12y^2-4y+1=0$$

Putting $Y=2y$, this becomes

$$Y^4+Y^3-3Y^2-2Y+1=(Y+1)(Y^3-3Y+1)=0 \tag{10}$$

Thus

$$Y+1=0, \qquad \text{This corresponds to } \cos\left(\frac{6\pi}{9}\right)=-\frac{1}{2}$$

or

$$(Y^3-3Y+1)=0$$

The above cubic equation can be solved by usig the so called "Cardano's method".
Letting $Y=s+t$, rewrite the above equation as

$$s^3+t^3+3(st-1)(s+t)+1=0$$

Choose s and t such that

$$
\begin{aligned}
s^3+t^3 &= -1 \qquad (11)\\
st &= 1 \quad \Rightarrow \quad s^3t^3=1 \qquad (12)
\end{aligned}
$$

Then s^3 and t^3 are solutions of

$$u^2+u+1=0$$

and we get

$$
u=\begin{cases}
\dfrac{-1+\sqrt{3}\,i}{2}=\omega=e^{\frac{2\pi}{3}i} \\[2ex]
\dfrac{-1-\sqrt{3}\,i}{2}=\omega^2=\overline{\omega}=e^{-\frac{2\pi}{3}i}.
\end{cases}
$$

We choose

$$s=\omega^{1/3}, \qquad t=\overline{\omega}^{1/3}$$

Then

$$\begin{aligned}
\cos\left(\frac{2\pi}{9}\right) &= \frac{s+t}{2} && (13)\\
&= \frac{\omega^{1/3}+\overline{\omega}^{1/3}}{2} && (14)\\
&= \frac{1}{2}\left(\sqrt[3]{\frac{-1+\sqrt{3}\,i}{2}}+\sqrt[3]{\frac{-1-\sqrt{3}\,i}{2}}\right) && (15)\\
&= 0.766044\ldots
\end{aligned}$$

$$\begin{aligned}
cos\left(\frac{4\pi}{9}\right) &= \frac{s\omega^2+t\omega}{2} && (16)\\
&= \frac{\omega^{7/3}+\overline{\omega}^{7/3}}{2} && (17)\\
&= \frac{\omega^{2/3}+\overline{\omega}^{2/3}}{2} && (18)\\
&= \frac{1}{2}\left[\left(\frac{-1+\sqrt{3}\,i}{2}\right)^{\frac{2}{3}}+\left(\frac{-1-\sqrt{3}\,i}{2}\right)^{\frac{2}{3}}\right] && (19)\\
&= 0.173648\ldots
\end{aligned}$$

$$\begin{aligned}
\cos\left(\frac{6\pi}{9}\right) &= \frac{\omega+\overline{\omega}}{2} && (20)\\
&= \cos\left(\frac{2\pi}{3}\right) && (21)\\
&= -\frac{1}{2} && (22)
\end{aligned}$$

$$\begin{aligned}
cos\left(\frac{8\pi}{9}\right) &= \frac{s\omega+t\omega^2}{2} && (23)\\
&= \frac{\omega^{4/3}+\overline{\omega}^{4/3}}{2} && (24)\\
&= \frac{1}{2}\left[\left(\frac{-1+\sqrt{3}\,i}{2}\right)^{\frac{4}{3}}+\left(\frac{-1-\sqrt{3}\,i}{2}\right)^{\frac{4}{3}}\right] && (25)\\
&= -0.939692\ldots && (26)
\end{aligned}$$

The meaning of the results is clear. Since

$$\cos\left(\frac{6\pi}{9}\right)=\cos\left(\frac{2\pi}{3}\right)=\frac{e^{2i\pi/3}+e^{-2i\pi/3}}{2}=\frac{\omega+\overline{\omega}}{2},$$

we obtain

$$\begin{aligned}
\cos\left(\frac{2k\pi}{9}\right) &= \cos\left(\frac{k}{3}\times\frac{6\pi}{9}\right)\\
&= \frac{\left(e^{2i\pi/3}\right)^{k/3}+\left(e^{-2i\pi/3}\right)^{k/3}}{2}\\
&= \frac{\omega^{k/3}+\overline{\omega}^{k/3}}{2}
\end{aligned}$$