The Values of $\cos\left(\frac{2\pi}{7}\right)$, $\cos\left(\frac{4\pi}{7}\right)$, and $\cos\left(\frac{6\pi}{7}\right)$

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Let

$$\theta = \frac{2\pi}{7},$$

then

$$x_k = e^{ik\theta} \qquad k = 0,\ 1,\ \cdots,\ 6$$

are the solutions of

$$x^7 - 1 = (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) = 0, \tag{1}$$

That is, the equation

$$x^4 + x^3 + x^2 + x + 1 = 0 \tag{2}$$

has the solutions of

$$e^{i\theta}, \quad e^{2i\theta}, \quad e^{3i\theta}, \quad e^{4i\theta} = e^{-3i\theta}, \quad e^{5i\theta} = e^{-2i\theta}, \quad e^{6i\theta} = e^{-i\theta}.$$

Therefore, (2) can be factorised as

$$\begin{aligned}
& x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 && (3)\\
&= \left(x - e^{i\theta}\right)\left(x - e^{-i\theta}\right)\left(x - e^{2i\theta}\right)\left(x - e^{-2i\theta}\right)\left(x - e^{3i\theta}\right)\left(x - e^{-3i\theta}\right)\\
&= \left(x^2 - 2x\cos\theta + 1\right)\left(x^2 - 2x\cos 2\theta + 1\right)\left(x^2 - 2x\cos 3\theta + 1\right)\\
&= x^6 - 2\left(\cos\theta + \cos 2\theta + \cos 3\theta\right)x^5\\
&\quad + \left(3 + 4\left(\cos\theta\cos 2\theta + \cos 2\theta\cos 3\theta + \cos 3\theta\cos\theta\right)\right)x^4\\
&\quad -4\left(\cos\theta + \cos 2\theta + \cos 3\theta + 2\cos\theta\cos 2\theta\cos 3\theta\right)x^3\\
&\quad + \left(3 + 4\left(\cos\theta\cos 2\theta + \cos 2\theta\cos 3\theta + \cos 3\theta\cos\theta\right)\right)x^2\\
&\quad -2\left(\cos\theta + \cos 2\theta + \cos 3\theta\right)x + 1 && (4)
\end{aligned}$$

Compairing the coefficients of (3) with those of (4), we find

$$\cos\theta + \cos 2\theta + \cos 3\theta = -\frac{1}{2} \tag{5}$$

$$\cos\theta\cos 2\theta + \cos 2\theta\cos 3\theta + \cos 3\theta\cos\theta = -\frac{1}{2} \tag{6}$$

$$\cos\theta\cos 2\theta\cos 3\theta = \frac{1}{8} \tag{7}$$

Next, we will consider the equation

$$\begin{aligned}
& (y - \cos\theta)(y - \cos 2\theta)(y - \cos 3\theta)\\
&= y^3 - (\cos\theta + \cos 2\theta + \cos 3\theta)y^2 + (\cos\theta + \cos 2\theta + \cos 3\theta)y - \cos\theta\cos 2\theta\cos 3\theta\\
&= y^3 + \frac{1}{2}y^2 - \frac{1}{2}y - \frac{1}{8}
\end{aligned}$$

Hence $\cos\theta$, $\cos 2\theta$ and $\cos 3\theta$ can be obtained by solving the cubic equation,

$$8y^3 + 4y^2 - 4y - 1 = 0$$

Putting $Y = 2y$, this becomes

$$Y^3 + Y^2 - 2Y - 1 = 0 \tag{8}$$

The Change of the variable

$$Y = z - \frac{1}{3}$$

gives the depressed cubic equation of z

$$z^3 - \frac{7}{3}z - \frac{7}{27} = 0.$$

Next, we express z as $z = s + t$, then

$$(s+t)^3 - \frac{7}{3}(s+t) - \frac{7}{27} = 0$$

or

$$s^3 + t^3 + \left(3st - \frac{7}{3}\right)(s+t) - \frac{7}{27} = 0$$

Choose s and t such that

$$s^3 + t^3 \;=\; \frac{7}{27} \tag{9}$$

$$3st \;=\; \frac{7}{3} \quad \Rightarrow \quad s^3t^3 = \frac{7^3}{9^3} \tag{10}$$

Hence s^3 and t^3 are the solutions of

$$u^2 - \frac{7}{3^3}u + \frac{7^3}{3^6} = 0$$

The solutions are

$$u = \frac{7}{2\cdot 3^3}\left(1 \pm 3\sqrt{3}\,i\right).$$

Here we choose

$$s \;=\; \frac{1}{3}\sqrt[3]{\frac{7}{2}\left(1 + 3\sqrt{3}\,i\right)} \tag{11}$$

$$t \;=\; \frac{1}{3}\sqrt[3]{\frac{7}{2}\left(1 - 3\sqrt{3}\,i\right)} = \overline{s} \tag{12}$$

$$\tag{13}$$

Then

$$\cos\left(\frac{2\pi}{7}\right) \;=\; \frac{Y}{2} \tag{14}$$

$$= \frac{1}{2}\left(z - \frac{1}{3}\right) \tag{15}$$

$$= \frac{1}{2}\left(s + t - \frac{1}{3}\right) \tag{16}$$

$$= \frac{1}{2}\left(s + \overline{s} - \frac{1}{3}\right) \tag{17}$$

$$= \frac{1}{6}\left(\sqrt[3]{\frac{7}{2}\left(1 + 3\sqrt{3}\,i\right)} + \sqrt[3]{\frac{7}{2}\left(1 - 3\sqrt{3}\,i\right)} - 1\right) \tag{18}$$

$$= 0.62348\ldots \tag{19}$$

The other two solutions are obtained by multiplying s and t by $\omega = \dfrac{-1+\sqrt{3}\,i}{2}$ or $\omega^2 = \dfrac{-1-\sqrt{3}\,i}{2} = \overline{\omega}$.
Namely,

$$\begin{aligned} \cos\left(\frac{4\pi}{7}\right) &= \frac{1}{2}\left(s\,\omega^2 + t\,\omega - \frac{1}{3}\right) && (20)\\ &= \frac{1}{2}\left(s\,\overline{\omega} + \overline{s}\,\omega - \frac{1}{3}\right) && (21)\\ &= -0.222520\ldots && (22) \end{aligned}$$

$$\begin{aligned} \cos\left(\frac{6\pi}{7}\right) &= \frac{1}{2}\left(s\,\omega + t\,\omega^2 - \frac{1}{3}\right) && (23)\\ &= \frac{1}{2}\left(s\,\omega + \overline{s\,\omega} - \frac{1}{3}\right) && (24)\\ &= -0.900968\ldots && (25) \end{aligned}$$

Addendum

Let's go back to

$$\left(x^2 - 2x\cos\theta + 1\right)\left(x^2 - 2x\cos 2\theta + 1\right)\left(x^2 - 2x\cos 3\theta + 1\right)$$

and rewrite this equation as

$$x^3\left(x + \frac{1}{x} - 2\cos\theta\right)\left(x + \frac{1}{x} - 2\cos 2\theta\right)\left(x + \frac{1}{x} - 2\cos 3\theta\right) \tag{26}$$

Substituting

$$x + \frac{1}{x} = Y$$

into (26), we obtain

$$x^3\left(Y - 2\cos\theta\right)\left(Y - 2\cos 2\theta\right)\left(Y - 2\cos 3\theta\right)$$

Next, rewrite (3) as

$$\begin{aligned} &x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 && (27)\\ &= x^3\left(x^3 + x^2 + x + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}\right) && (28) \end{aligned}$$

Express the equation in parentheses by means of Y.
Since

$$\begin{aligned} Y^3 &= x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) && (29)\\ Y^2 &= x^2 + \frac{1}{x^2} + 2 && (30)\\ Y &= x + \frac{1}{x} && (31) \end{aligned}$$

we get the relation.

$$
\begin{aligned}
&Y^3+Y^2+bY+c \\
&=\left(x^3+\frac{1}{x^3}\right)+\left(x^2+\frac{1}{x^2}\right)+(3+b)\left(x+\frac{1}{x}\right)+(2+c) \\
&=x^3+x^2+x+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}
\end{aligned}
$$

This determines the parameters b and c as

$$
b=-2, \quad c=-1
$$

Hence, we finally obtain

$$
\begin{aligned}
(Y-2\cos\theta)(Y-2\cos 2\theta)(Y-2\cos 3\theta) &= Y^3+Y^2-2Y-1 \\
&= 0
\end{aligned}
$$