The Values of $\cos\left(\frac{\pi}{5}\right)$, $\cos\left(\frac{2\pi}{5}\right)$, $\cos\left(\frac{3\pi}{5}\right)$, and $\cos\left(\frac{4\pi}{5}\right)$

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Let

$$\theta = \frac{2\pi}{10} = \frac{\pi}{5},$$

then

$$x_k = e^{ik\theta} \qquad k = 0,\ 1,\ 2,\ \cdots,\ 9$$

are the solutions of

$$x^{10} - 1 = 0$$

$$\begin{aligned} x^{10} - 1 &= (x^5 - 1)(x^5 + 1) \\ &= (x-1)(x^4 + x^3 + x^2 + x + 1)(x+1)(x^4 - x^3 + x^2 - x + 1) \end{aligned}$$

Hence

$$x = \pm 1, \quad x^4 + x^3 + x^2 + x + 1 = 0, \quad \text{or} \quad \mathrm{x}^4 - \mathrm{x}^3 + \mathrm{x}^2 - \mathrm{x} + 1 = 0$$

1 Solutions of $\pm 1 = 0$

$$\begin{aligned} x &= 1 \ \Rightarrow \cos 0 = 1 \\ x &= -1 \ \Rightarrow \cos\pi = -1 \end{aligned}$$

2 Solutions of $x^4 + x^3 + x^2 + x + 1 = 0$

$$x_{2m} = e^{2im\theta} = e^{2\pi im/5} \qquad m = 1,\ 2,\ 3,\ 4$$

Hence

$$\begin{aligned} & x^4 + x^3 + x^2 + x + 1 \\ &= \left(x - e^{2\pi i/5}\right)\left(x - e^{4\pi i/5}\right)\left(x - e^{-4\pi i/5}\right)\left(x - e^{-2\pi i/5}\right) \\ &= \left(x^2 - 2\cos\left(\frac{2\pi}{5}\right) + 1\right)\left(x^2 - 2\cos\left(\frac{4\pi}{5}\right) + 1\right) \\ &= x^2\left(x + \frac{1}{x} - 2\cos\left(\frac{2\pi}{5}\right)\right)\left(x + \frac{1}{x} - 2\cos\left(\frac{4\pi}{5}\right)\right) \\ &= x^2\left(x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2}\right) \end{aligned}$$

Let

$$Y = x + \frac{1}{x}$$

Then

$$\left(Y - 2\cos\left(\frac{2\pi}{5}\right)\right)\left(Y - 2\cos\left(\frac{4\pi}{5}\right)\right) = Y^2 + Y - 1 = 0$$

Thus

$$\begin{aligned}
\cos\left(\frac{2\pi}{5}\right) &= \frac{Y}{2} = \frac{-1+\sqrt{5}}{4} \\
\cos\left(\frac{4\pi}{5}\right) &= \frac{Y}{2} = \frac{-1-\sqrt{5}}{4}
\end{aligned}$$

3 Solutions of $x^4 - x^3 + x^2 - x + 1 = 0$

Solutions of

$$x^4 - x^3 + x^2 - x + 1 = \frac{x^5+1}{x+1} = 0$$

are

$$x_{2m+1} = e^{i(2m+1)\theta} = e^{(2m+1)\pi i/5} \qquad m = 0,\ 1,\ 3,\ 4$$

Hence

$$\begin{aligned}
&x^4 - x^3 + x^2 - x + 1 \\
&= \left(x - e^{\pi i/5}\right)\left(x - e^{\pi i/5}\right)\left(x - e^{3\pi i/5}\right)\left(x - e^{-3\pi i/5}\right) \\
&= \left(x^2 - 2x\cos\left(\frac{\pi}{5}\right) + 1\right)\left(x^2 - 2x\cos\left(\frac{3\pi}{5}\right) + 1\right) \\
&= x^2\left(x + \frac{1}{x} - 2\cos\left(\frac{\pi}{5}\right)\right)\left(x + \frac{1}{x} - 2\cos\left(\frac{3\pi}{5}\right)\right) \\
&x^2\left(x^2 - x + 1 - \frac{1}{x} + \frac{1}{x^2}\right)
\end{aligned}$$

Let

$$Y = x + \frac{1}{x}$$

Then

$$\left(Y - 2\cos\left(\frac{\pi}{5}\right)\right)\left(Y - 2\cos\left(\frac{3\pi}{5}\right)\right) = Y^2 - Y + 1 = 0$$

Hence

$$\begin{aligned}
\cos\left(\frac{\pi}{5}\right) &= \frac{Y}{2} = \frac{1+\sqrt{5}}{4} \qquad (1) \\
\cos\left(\frac{3\pi}{5}\right) &= \frac{Y}{2} = \frac{1-\sqrt{5}}{4} \qquad (2)
\end{aligned}$$

4 Summary

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\begin{aligned}
\cos\left(\frac{\pi}{5}\right) &= \frac{1+\sqrt{5}}{4} \\
\cos\left(\frac{2\pi}{5}\right) &= \frac{-1+\sqrt{5}}{4} \\
\cos\left(\frac{3\pi}{5}\right) &= \frac{1-\sqrt{5}}{4} \\
\cos\left(\frac{4\pi}{5}\right) &= \frac{-1-\sqrt{5}}{4}
\end{aligned}
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