The Algebraic Values of $\cos 36^\circ$ and $\cos 72^\circ$

Shoichi MIdorikawa

Consider the following quintic equation

$$x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1) = 0. \tag{1}$$

Define θ by

$$\theta = \frac{2\pi}{5} = 72^\circ,$$

then the solutions of (1) are written as

$$x_k = e^{ik\theta} \qquad k = 0,\ 1,\ \cdots,\ 4$$

Hence the solutions of the equation

$$x^4 + x^3 + x^2 + x + 1 = 0 \tag{2}$$

are

$$e^{i\theta}, \quad e^{2i\theta}, \quad e^{3i\theta} = e^{-2i\theta}, \quad e^{4i\theta} = e^{-i\theta},$$

that is, (2) can be factorised as

$$\begin{aligned} & x^4 + x^3 + x^2 + x + 1 && (3)\\ &= \left(x - e^{i\theta}\right)\left(x - e^{-i\theta}\right)\left(x - e^{2i\theta}\right)\left(x - e^{-2i\theta}\right) \\ &= \left(x^2 - 2x\cos\theta + 1\right)\left(x^2 - 2x\cos 2\theta + 1\right) \\ &= x^4 - 2\left(\cos\theta + \cos 2\theta\right)x^3 + 2\left(1 + 2\cos\theta\cos 2\theta\right)x^2 - 2\left(\cos\theta + \cos 2\theta\right)x + 1 && (4) \end{aligned}$$

Compairing the coefficients of (3) with those of (4), we find

$$\cos\theta + \cos 2\theta = -\frac{1}{2} \tag{5}$$

$$\cos\theta\cos 2\theta = -\frac{1}{4} \tag{6}$$

Hence $\cos\theta$ and $\cos 2\theta$ can be obtained by solving the quadratic equation,

$$y^2 + \frac{1}{2}y - \frac{1}{4} = 0.$$

The solutions are

$$y = \frac{-1 \pm \sqrt{5}}{4}$$

Since $\cos\theta > 0$ and $\cos 2\theta < 0$, we have

$$\cos\theta = \frac{-1 + \sqrt{5}}{4} \tag{7}$$

$$\cos 2\theta = \frac{-1 - \sqrt{5}}{4} \tag{8}$$

Note that $\theta = 72^\circ$ and $2\theta = 144^\circ = 180^\circ - 36^\circ$, which lead to

$$\cos 72^\circ = \frac{-1 + \sqrt{5}}{4} \tag{9}$$

$$\cos 36^\circ = -\cos 144^\circ = \frac{1 + \sqrt{5}}{4} \tag{10}$$