

The Values of $\cos\left(\frac{2\pi}{7}\right)$, $\cos\left(\frac{4\pi}{7}\right)$, and $\cos\left(\frac{6\pi}{7}\right)$

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Let

$$\theta = \frac{2\pi}{7},$$

then

$$x_k = e^{ik\theta} \quad k = 0, 1, \dots, 6$$

are the solutions of

$$x^7 - 1 = (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) = 0, \quad (1)$$

That is, the equation

$$x^4 + x^3 + x^2 + x + 1 = 0 \quad (2)$$

has the solutions of

$$e^{i\theta}, \quad e^{2i\theta}, \quad e^{3i\theta}, \quad e^{4i\theta} = e^{-3i\theta}, \quad e^{5i\theta} = e^{-2i\theta}, \quad e^{6i\theta} = e^{-i\theta}.$$

Therefore, (2) can be factorised as

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \quad (3)$$

$$= (x - e^{i\theta})(x - e^{-i\theta})(x - e^{2i\theta})(x - e^{-2i\theta})(x - e^{3i\theta})(x - e^{-3i\theta})$$

$$= (x^2 - 2x \cos \theta + 1)(x^2 - 2x \cos 2\theta + 1)(x^2 - 2x \cos 3\theta + 1)$$

$$= x^6 - 2(\cos \theta + \cos 2\theta + \cos 3\theta)x^5$$

$$+ (3 + 4(\cos \theta \cos 2\theta + \cos 2\theta \cos 3\theta + \cos 3\theta \cos \theta))x^4$$

$$- 4(\cos \theta + \cos 2\theta + \cos 3\theta + 2 \cos \theta \cos 2\theta \cos 3\theta)x^3$$

$$+ (3 + 4(\cos \theta \cos 2\theta + \cos 2\theta \cos 3\theta + \cos 3\theta \cos \theta))x^2$$

$$- 2(\cos \theta + \cos 2\theta + \cos 3\theta)x + 1 \quad (4)$$

Comparing the coefficients of (3) with those of (4), we find

$$\cos \theta + \cos 2\theta + \cos 3\theta = -\frac{1}{2} \quad (5)$$

$$\cos \theta \cos 2\theta + \cos 2\theta \cos 3\theta + \cos 3\theta \cos \theta = -\frac{1}{2} \quad (6)$$

$$\cos \theta \cos 2\theta \cos 3\theta = \frac{1}{8} \quad (7)$$

Next, we will consider the equation

$$(y - \cos \theta)(y - \cos 2\theta)(y - \cos 3\theta)$$

$$= y^3 - (\cos \theta + \cos 2\theta + \cos 3\theta)y^2 + (\cos \theta + \cos 2\theta + \cos 3\theta)y - \cos \theta \cos 2\theta \cos 3\theta$$

$$= y^3 + \frac{1}{2}y^2 - \frac{1}{2}y - \frac{1}{8}$$

Hence $\cos \theta$, $\cos 2\theta$ and $\cos 3\theta$ can be obtained by solving the cubic equation,

$$8y^3 + 4y^2 - 4y - 1 = 0$$

Putting $Y = 2y$, this becomes

$$Y^3 + Y^2 - 2Y - 1 = 0 \quad (8)$$

The Change of the variable

$$Y = z - \frac{1}{3}$$

gives the depressed cubic equation of z

$$z^3 - \frac{7}{3}z - \frac{7}{27} = 0.$$

Next, we express z as $z = s + t$, then

$$(s + t)^3 - \frac{7}{3}(s + t) - \frac{7}{27} = 0$$

or

$$s^3 + t^3 + \left(3st - \frac{7}{3}\right)(s + t) - \frac{7}{27} = 0$$

Choose s and t such that

$$s^3 + t^3 = \frac{7}{27} \quad (9)$$

$$3st = \frac{7}{3} \Rightarrow s^3 t^3 = \frac{7^3}{9^3} \quad (10)$$

Hence s^3 and t^3 are the solutions of

$$u^2 - \frac{7}{3^3}u + \frac{7^3}{3^6} = 0$$

The solutions are

$$u = \frac{7}{2 \cdot 3^3} \left(1 \pm 3\sqrt{3}i\right).$$

Here we choose

$$s = \frac{1}{3} \sqrt[3]{\frac{7}{2} \left(1 + 3\sqrt{3}i\right)} \quad (11)$$

$$t = \frac{1}{3} \sqrt[3]{\frac{7}{2} \left(1 - 3\sqrt{3}i\right)} = \bar{s} \quad (12)$$

$$(13)$$

Then

$$\cos\left(\frac{2\pi}{7}\right) = \frac{Y}{2} \quad (14)$$

$$= \frac{1}{2} \left(z - \frac{1}{3}\right) \quad (15)$$

$$= \frac{1}{2} \left(s + t - \frac{1}{3}\right) \quad (16)$$

$$= \frac{1}{2} \left(s + \bar{s} - \frac{1}{3}\right) \quad (17)$$

$$= \frac{1}{6} \left(\sqrt[3]{\frac{7}{2} \left(1 + 3\sqrt{3}i\right)} + \sqrt[3]{\frac{7}{2} \left(1 - 3\sqrt{3}i\right)} - 1\right) \quad (18)$$

$$= 0.62348 \dots \quad (19)$$

The other two solutions are obtained by multiplying s and t by $\omega = \frac{-1 + \sqrt{3}i}{2}$ or

$$\omega^2 = \frac{-1 - \sqrt{3}i}{2} = \bar{\omega}.$$

Namely,

$$\cos\left(\frac{4\pi}{7}\right) = \frac{1}{2}\left(s\omega^2 + t\omega - \frac{1}{3}\right) \quad (20)$$

$$= \frac{1}{2}\left(s\bar{\omega} + \bar{s}\omega - \frac{1}{3}\right) \quad (21)$$

$$= -0.222520\dots \quad (22)$$

$$\cos\left(\frac{6\pi}{7}\right) = \frac{1}{2}\left(s\omega + t\omega^2 - \frac{1}{3}\right) \quad (23)$$

$$= \frac{1}{2}\left(s\omega + \bar{s}\bar{\omega} - \frac{1}{3}\right) \quad (24)$$

$$= -0.900968\dots \quad (25)$$