Gaussian Integral Using Residue Theorem
— The Symmetric Contour —

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1 The Original Contour

The residue theorem can be applied to the Gaussian integral[1] by finding the function $f_a(z)$ which satisfies the relation

$$f_a(z) - f_a(z+\tau) = e^{-z^2/2} \tag{1}$$

around the rectangular contour depicted in Fig.1. Here, the index a denotes the symbol for "asymmetric".

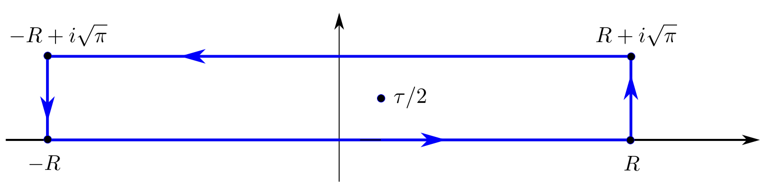

Figure 1: Original contour

Since I have not read the Kneser's original paper[1], my proof may be different from that of the previous work, although the conclusion is obviously the same.

Let us determine $f_a(z)$ in the following way. If the function $f_a(z)$ is proportional to $e^{-z^2/2}$, i.e.,

$$f_a(z) \propto e^{-z^2/2} \tag{2}$$

then

$$f_a(z+\tau) \propto e^{-(z+\tau)^2/2} = e^{-\tau z} e^{-\tau^2/2} e^{-z^2/2}. \tag{3}$$

So it is natural to assume

$$f_a(z+\tau) = e^{-\tau z} e^{-\tau^2/2} f_a(z) \tag{4}$$

Solving the simultaneous equations involving (1) and (4), we have

$$\begin{aligned} f_a(z) &= \frac{e^{-z^2/2}}{1 - e^{-\tau z} e^{-\tau^2/2}} &(5)\\ f_a(z+\tau) &= \frac{e^{-(z+\tau)^2/2}}{1 - e^{-\tau z} e^{-\tau^2/2}} &(6)\\ &= \frac{e^{-z^2/2} e^{-\tau z} e^{-\tau^2/2}}{1 - e^{-\tau z} e^{-\tau^2/2}} &(7)\\ &= -\frac{e^{-z^2/2}}{1 - e^{\tau z} e^{\tau^2/2}} &(8) \end{aligned}$$

From (5), we get the following expression of $f_a(z+\tau)$

$$f_a(z+\tau) = \frac{e^{-(z+\tau)^2/2}}{1 - e^{-\tau(z+\tau)} e^{-\tau^2/2}} \tag{9}$$

Since (9) should be equal to (6)

$$\frac{e^{-(z+\tau)^2/2}}{1 - e^{-\tau z} e^{-\tau^2/2}} = \frac{e^{-(z+\tau)^2/2}}{1 - e^{-\tau z} e^{-\tau^2} e^{-\tau^2/2}} \tag{10}$$

which leads to

$$e^{\tau^2} = 1. \tag{11}$$

In other words,

$$\tau^2 = 2\pi i n \text{ for integer } n$$

The simple solution of this is in the case of $n = 1$, that is, $\tau^2 = 2\pi i$ and

$$\tau = \sqrt{2\pi}\, e^{\pi i/4} = \sqrt{\pi}(1+i).$$

We also have the following relation.

$$e^{-\tau^2/2} = e^{-\pi i} = -1, \tag{12}$$

By substituting (12) into (5) and (8), we find

$$\begin{aligned} f_a(z) &= \frac{e^{-z^2/2}}{1 + e^{-\tau z}}, &(13)\\ f_a(z+\tau) &= -\frac{e^{-z^2/2}}{1 + e^{\tau z}}. &(14) \end{aligned}$$

We can verify that the equation (1) holds.

$$\begin{aligned} f_a(z) - f_a(z+\tau) &= \frac{e^{-z^2/2}}{1+e^{-\tau z}} + \frac{e^{-z^2/2}e^{-\tau z}}{1+e^{-\tau z}} \\ &= e^{-z^2/2} \end{aligned}$$

$1+e^{-\tau z} = 0$ at $\tau z = \pi i$. Since $\tau^2 = 2\pi i$, $\tau z = \dfrac{\tau^2}{2}$. Thus $f_a(z)$ has a pole at $z = \dfrac{\tau}{2}$

$$\frac{d}{dz}(1+e^{-\tau z})\Big|_{z=\tau/2} = -\tau e^{-\tau^2/2} = \tau$$

The residue of $f_a(z)$ at $z = \tau/2$

$$\frac{e^{-\tau^2/8}}{\tau} = \frac{e^{-\pi i/4}}{\sqrt{2\pi}\, e^{\pi i/4}} = \frac{e^{-\pi i/2}}{\sqrt{2\pi}} = \frac{-i}{\sqrt{2\pi}}$$

Thus,

$$\oint f_a(z)\, dz = 2\pi i \times \frac{-i}{\sqrt{2\pi}} \tag{15}$$

As $R \to \infty$ the value of $|f(z)|$ tends to 0 uniformly along the left and right fringes of the contour, we obtain

$$\int_{-\infty}^{\infty} f_a(x)\, dx + \int_{\infty+i\sqrt{\pi}}^{-\infty+i\sqrt{\pi}} f_a(z)\, dz = \sqrt{2\pi}$$

Therefore

$$\begin{aligned} \sqrt{2\pi} &= \int_{-\infty}^{\infty} f_a(x)\, dx - \int_{-\infty+i\sqrt{\pi}}^{\infty+i\sqrt{\pi}} f_a(z)\, dz \\ &= \int_{-\infty}^{\infty} f_a(x)\, dx - \int_{-\infty}^{\infty} f_a(x+i\sqrt{\pi})\, dx \quad (z = x+i\sqrt{\pi}) \\ &= \int_{-\infty}^{\infty} f_a(x)\, dx - \int_{-\infty}^{\infty} f_a(x-\sqrt{\pi}+\tau)\, dx \quad (\tau = \sqrt{\pi}+i\sqrt{\pi}) \\ &= \int_{-\infty}^{\infty} f_a(x)\, dx - \int_{-\infty}^{\infty} f_a(x+\tau)\, dx \quad (x-\sqrt{\pi} \to x) \\ &= \int_{-\infty}^{\infty} (f_a(x) - f_a(x+\tau))\, dx \\ &= \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \end{aligned}$$

Hence,

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi} \tag{16}$$

2 The Symmetric Contour

Now consider the function $f_s(z)$ which satisfies the "symmetric" contour.

$$f_s\left(z - \frac{\tau}{2}\right) - f_s\left(z + \frac{\tau}{2}\right) = e^{-z^2/2}$$

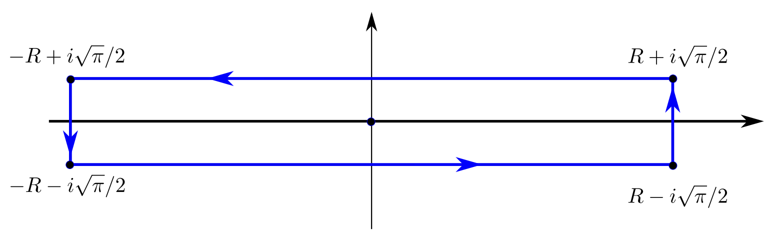

Figure 2: The Symmetric Path

Comparing this with (1)

$$f_a(z) - f_a(z + \tau) = e^{-z^2/2}$$

we find that

$$f_s\left(z - \frac{\tau}{2}\right) = f_a(z) = \frac{e^{-z^2/2}}{1 + e^{-\tau z}}, \tag{17}$$

$$f_s\left(z + \frac{\tau}{2}\right) = f_a(z + \tau) = -\frac{e^{-z^2/2}}{1 + e^{\tau z}}. \tag{18}$$

Namely

$$f_s(z) = f_a\left(z + \frac{\tau}{2}\right) = \frac{e^{-(z+\tau/2)^2/2}}{1 + e^{-\tau(z+\tau/2)}} = \frac{e^{-(z+\tau/2)^2/2}}{1 - e^{-\tau z}}$$

Since $1 - e^{-\tau z} = 0$ at $\tau z = 0$, $f_s(z)$ has a pole at $z = 0$

$$\frac{d}{dz}(1 - e^{-\tau z})\Big|_{z=0} = \tau e^{-\tau z}\Big|_{z=0} = \tau$$

The residue of $f_s(z)$ at $z = 0$ is

$$\frac{e^{-\tau^2/8}}{\tau} = \frac{e^{-\pi i/4}}{\sqrt{2\pi}\, e^{\pi i/4}} = \frac{e^{-\pi i/2}}{\sqrt{2\pi}} = \frac{-i}{\sqrt{2\pi}}$$

Therefore

$$\oint f_s(z)\, dz = \sqrt{2\pi} \tag{19}$$

As $R \to \infty$ the value of $|f(z)|$ tends to 0 uniformly along the left and right fringes of the contour, we obtain

$$\int_{-\infty - i\sqrt{\pi}/2}^{\infty - i\sqrt{\pi}/2} f_s(z)\, dz + \int_{\infty + i\sqrt{\pi}/2}^{-\infty + i\sqrt{\pi}/2} f_s(x)\, dx = 2\pi i \times \frac{-i}{\sqrt{2\pi}}$$

$$\begin{aligned}
\sqrt{2\pi} &= \int_{-\infty - i\sqrt{\pi}/2}^{\infty - i\sqrt{\pi}/2} f_s(z)\, dz - \int_{-\infty + i\sqrt{\pi}/2}^{\infty + i\sqrt{\pi}/2} f_s(z)\, dz \\
&= \int_{-\infty}^{\infty} f_s\left(x - i\frac{\sqrt{\pi}}{2}\right) dx - \int_{-\infty}^{\infty} f_s\left(x + i\frac{\sqrt{\pi}}{2}\right) dx \quad \left(z = x \mp i\frac{\sqrt{\pi}}{2}\right) \\
&= \int_{-\infty}^{\infty} f_s\left(x + \frac{\sqrt{\pi}}{2} - \frac{\tau}{2}\right) dx - \int_{-\infty}^{\infty} f_s\left(x - \frac{\sqrt{\pi}}{2} + \frac{\tau}{2}\right) dx \quad (\tau = \sqrt{\pi} + i\sqrt{\pi}) \\
&= \int_{-\infty}^{\infty} f_s\left(x - \frac{\tau}{2}\right) dx - \int_{-\infty}^{\infty} f_s\left(x + \frac{\tau}{2}\right) dx \quad \left(x \pm \frac{\sqrt{\pi}}{2} \to x\right) \\
&= \int_{-\infty}^{\infty} \left(f_s\left(x - \frac{\tau}{2}\right) - f_s\left(x + \frac{\tau}{2}\right)\right) dx \\
&= \int_{-\infty}^{\infty} e^{-x^2/2}\, dx
\end{aligned}$$

Hence,

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi} \tag{20}$$

References

[1] H. Kneser, Funktionentheorie, Vandenhoeck and Ruprecht, 1958.
For reviews, see
https://math.stackexchange.com/questions/34767/int-infty-infty-e-x2-dx-with-complex-analysis
Keith Conrad, "The Gaussian Integral"