The Alternating Group A_n Is Generated by 3-Cycles

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The symmetric group S_n is the group consisting of all permutations of the n symbols $\{1,\ 2,\ 3,\ \cdots,\ n\}$. The number of elements in this group is $n!$. These elements can be expressed as products of transpositions. Among them, the subset consisting of products of an even number of transpositions, that is, the even permutations, forms the alternating group A_n, whose number of elements is $\dfrac{n!}{2}$.

Every element of A_n can be written using pairs of transpositions. When the product of two transpositions is illustrated by an "Amidakuji(ladder lottery)" diagram, there are two cases: the two transpositions contain a common number(Figure 1), or all the numbers are distinct(Figure 2).

We show that, for $n \geq 3$, the alternating group A_n is generated by 3-cycles. For this purpose, it is enough to show that a product of transpositions can be expressed as a 3-cycle or as a product of 3-cycles. There are two conventions for composing permutations. Here, the operation of first applying the permutation π_1 and then applying the permutation π_2 is written as $\pi_2\pi_1$.

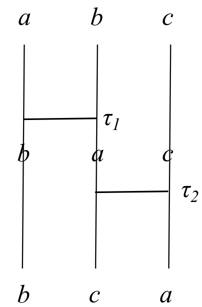

Figure 1: $\tau_2\tau_1$

The case where two transpositions contain the same number *a*

Let the two transpositions be

$$\tau_1 = (a\ b), \quad \tau_2 = (a\ c) \tag{1}$$

Then

$$\begin{aligned} \tau_2\tau_1 &= (a\ c)(a\ b) \\ &= (a\ b\ c) \end{aligned}$$

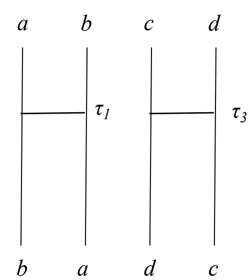

Figure 2: $\tau_3\tau_1$

The case where two transpositions do not contain a common number

In this case, put $\tau_3 = (c\ d)$ and examine $\tau_3\tau_1$. Since the result in Figure 2 can also be reproduced as in Figure 3,

$$\begin{aligned} \tau_3\tau_1 &= \tau_5\tau_4\tau_2\tau_1 \\ &= (a\ c)(a\ d)(a\ c)(a\ b) \\ &= \big\{(a\ c)(a\ d)\big\}\big\{(a\ c)(a\ b)\big\} \\ &= (a\ d\ c)(a\ b\ c) \end{aligned}$$

$$\therefore\ (c\ d)(a\ b) = (a\ d\ c)(a\ b\ c)$$

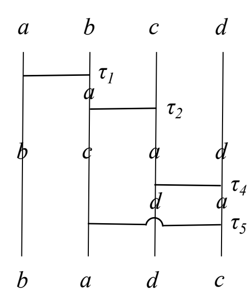

Figure 3: Another representation of Figure 2

Thus, every product of transpositions can be expressed as a 3-cycle or as a product of 3-cycles. Therefore, for $n \geq 3$, the group A_n is generated by 3-cycles.