Algebra of Random Variables

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1 Distribution of a Random Variable

Let X and $Y$are mutually independent random variables, and their probability density functions (PDFs) are given by $f_X(x)$ and $f_Y(y)$, respectively. Suppose Z is another random variable which is a function of X and Y.

$$Z = \varphi(X, Y) \tag{1}$$

Then, the probaboloty density function of $f_Z(z)$ is

$$f_Z(z) = \int \delta(z - \varphi(x, y)) f_X(x) f_Y(y)\, dx\, dy. \tag{2}$$

Note that

$$f_Z(z) \geq 0$$

and

$$\begin{aligned} \int f_Z(z)\, dz &= \int \delta(z - \varphi(x, y)) f_X(x) f_Y(y)\, dx\, dy\, dz \\ &= \int f_X(x)\, dx \int f_Y(y)\, dy \\ &= 1 \end{aligned}$$

The integral in (2) with respect to x can easily be done thanks to the delta function $\delta(z - \varphi(x, y))$. For this purpose we rewrite the delta dunction. Let α_i be the i-th solution of $z = \varphi(\alpha_i, y)$, which means that α is a function of y and z. If $|x - \alpha_i| \ll 1$

$$z - \varphi(x, y) \approx -\frac{\partial \varphi(\alpha_i, y)}{\partial x}(x - \alpha_i),$$

we obtain the relation

$$\begin{aligned} \delta(z - \varphi(x, y)) &= \sum_i \delta\left(\frac{\partial \varphi(\alpha_i, y)}{\partial x}(x - \alpha_i)\right) \\ &= \sum_i \frac{1}{|\partial \varphi(\alpha_i, y)/\partial x|} \delta(x - \alpha_i) \end{aligned} \tag{3}$$

Substitution (3) into (2) leads to

$$f_Z(z) = \sum_i \int \frac{1}{|\partial\varphi(\alpha_i, y)/\partial x|} \delta(x - \alpha_i) f_X(x) f_Y(y)\, dx\, dy \tag{4}$$

Integrate this equation with respsect to x, we finally find

$$f_Z(z) = \sum_i \int \frac{1}{|\partial\varphi(\alpha_i, y)/\partial x|} f_X(\alpha_i) f_Y(y)\, dy \tag{5}$$

2 Expectation Values of Random Variables

The expectation value of $\xi(X)$ in terms of $F_X(x)$ is defined by

$$E[\xi(X)] = \int \xi(x) f_X(x)\, dx \tag{6}$$

2.1 $Z = \xi(X) + \eta(Y)$

Suppose $\xi(X)$ is a function of X, and $\eta(Y)$ is a function of Y. Furthermore, A random Variable Z is a sum of $\xi(X)$ and $\eta(Y)$.

$$Z = \xi(X) + \eta(Y) \tag{7}$$

Then, we have

$$f_Z(z) = \int \delta(z - \xi(x) - \eta(y)) f_X(x) f_Y(y)\, dx\, dy \tag{8}$$

The expectation value of Z is

$$\begin{aligned}
E[Z] &= \int z f_Z(z)\, dz \\
&= \int z\, \delta(z - \xi(x) - \eta(y)) f_X(x) f_Y(y)\, dx\, dy\, dz \\
&= \int (\xi(x) + \eta(y)) f_X(x) f_Y(y)\, dx\, dy \\
&= \int \xi(x) f_X(x)\, dx \int f_Y(y)\, dy + \int f_X(x)\, dx \int \eta(y) f_Y(y)\, dy \\
&= \int \xi(x) f_X(x)\, dx + \int \eta(y) f_Y(y)\, dy
\end{aligned}$$

Thus we find that

$$E[\xi(X) + \eta(Y)] = E[\xi(X)] + E[\eta(Y)] \tag{9}$$

As a special case, consider the case of

$$\xi(X) = aX, \quad \text{and} \quad \eta(Y) = bY. \tag{10}$$

Then (9) becomes

$$E[aX + bY] = aE[X] + bE[Y]. \tag{11}$$

2.2 $Z = \xi(X)\eta(Y)$

Next, we consider the case that Z is a multiple of $\xi(X)$ and $\eta(Y)$, namely,

$$Z = \xi(X)\eta(Y) \tag{12}$$

Then we have

$$f_Z(z) = \int \delta(z - \xi(x)\eta(y)) f_X(x) f_Y(y)\, dx\, dy \tag{13}$$

The expectation value of Z is

$$E[Z] = \int z f_Z(z)\, dz \tag{14}$$

$$= \int z\delta(z - \xi(x)\eta(y)) f_X(x) f_Y(y)\, dx\, dy\, dz \tag{15}$$

$$= \int \xi(x) f_X(x)\, dx \cdot \int \eta(y) f_Y(y)\, dy \tag{16}$$

Thus we obtain

$$E[\xi(X)\eta(Y)] = E[\xi(X)] \cdot E[\eta(Y)] \tag{17}$$

As the special cases, we consider the two typical cases, $Z = XY$ and $Z = X/Y$. In case of $\xi(X) = X$ and $\eta(Y) = Y$, we have $Z = XY$ and

$$E[XY] = E[X] \cdot E[Y] \tag{18}$$

In case of $\xi(X) = X$ and $\eta(Y) = 1/Y$, we have $Z = X/Y$ and

$$E\left[\frac{X}{Y}\right] = E[X] \cdot E\left[\frac{1}{Y}\right] \tag{19}$$

2.3 $Z = X^Y$

$$f_Z(z) = \int \delta(z - x^y) f_X(x) f_Y(y)\, dx\, dy \tag{20}$$

$$= \int \frac{1}{|y\, x^{y-1}|} \delta\left(x - z^{1/y}\right) f_X(x) f_Y(y)\, dx\, dy \tag{21}$$

$$f_Z(z) = \int \frac{1}{|y\, z^{1-1/y}|} f_X\left(z^{1/y}\right) f_Y(y)\, dy \tag{22}$$

$$\begin{aligned} E[Z] &= \int z f_Z(z)\, dz && (23)\\ &= \int z\delta(z - x^y) f_X(x) f_Y(y)\, dx\, dy\, dz && (24)\\ &= \int x^y f_X(x) f_Y(y)\, dx\, dy && (25)\\ &= \int e^{y \ln x} f_X(x) f_Y(y)\, dx\, dy && (26)\\ &= \int \left(\sum_{n=0}^{\infty} \frac{1}{n!} y^n \ln^n x \right) f_X(x) f_Y(y)\, dx\, dy && (27) \end{aligned}$$

$$\begin{aligned} \therefore E[X^Y] &= \int x^y f_X(x) f_Y(y)\, dx\, dy && (28)\\ &= \sum_{n=0}^{\infty} \frac{1}{n!} E[\ln^n x] \cdot E[y^n] && (29) \end{aligned}$$

3 Variance of Random Variables

The variance of X is defined by

$$\mathrm{Var}(X) = \int (x - E[X])^2 f_X(x)\, dx \tag{30}$$

This can be rewrittten as

$$\begin{aligned} \mathrm{Var}(X) &= \int \left(x^2 - 2E[X]x + E[x]^2\right) f_X(x)\, dx && (31)\\ &= \int x^2 f_X(x)\, dx - 2E[X] \int x f_X(x)\, dx + E[x]^2 \int f_X(x)\, dx && (32) \end{aligned}$$

Therefore

$$\begin{aligned} \mathrm{Var}(X) &= E\left[(X - E[X])^2\right] && (33)\\ &= E[X^2] - E[X]^2 && (34) \end{aligned}$$

3.1 $Z = \xi(X) + \eta(Y)$

From the definition,

$$\mathrm{Var}(Z) = E[Z^2] - E[Z]^2 \tag{35}$$

Here,

$$E[Z^2] = E\left[(\xi(X)+\eta(Y))^2\right] \tag{36}$$
$$= E[\xi(X)^2] + 2E[\xi(X)]E[\eta(Y)] + E[\eta(Y)]^2 \tag{37}$$

$$E[Z]^2 = E[\xi(X)+\eta(Y)]^2 \tag{38}$$
$$= (E[\xi(X)] + E[\eta(Y)])^2 \tag{39}$$
$$= E[\xi(X)]^2 + 2E[\xi(X)]E[\eta(Y)] + E[\eta(Y)]^2 \tag{40}$$

By substituting (37) and (40) into (35), we obtain

$$\mathrm{Var}(Z) = \mathrm{Var}(\xi(X)) + \mathrm{Var}(\eta(Y)) \tag{41}$$

Especially, in the case of $\xi(X) = aX$ and $\eta(Y) = bY$, we have

$$\mathrm{Var}(Z) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) \tag{42}$$

3.2 $Z = \xi(X)\eta(Y)$

Let

$$Z = \xi(X)\eta(Y)$$

then

$$\mathrm{Var}(Z) = \int (Z - E[Z])^2 f_Z(z)\,dz \tag{43}$$
$$= E[Z^2] - E[Z^2] \tag{44}$$

Thus,

$$\mathrm{Var}(\xi(X)\eta(Y)) = E\left[\xi(X)^2\eta(Y)^2\right] - E[\xi(X)\eta(Y)]^2 \tag{45}$$

Here,

$$E\left[\xi(X)^2\eta(Y)^2\right] = E\left[\xi(X)^2\right]E\left[\eta(Y)^2\right] \tag{46}$$
$$E[\xi(X)\eta(Y)]^2 = E[\xi(X)]^2E[\eta(Y)]^2 \tag{47}$$

By substituting (46) and (47) into (44), we obtain

$$\mathrm{Var}(\xi(X)\eta(Y)) \tag{48}$$
$$= E\left[\xi(X)^2\right]E\left[\eta(Y)^2\right] - E[\xi(X)]^2E[\eta(Y)]^2 \tag{49}$$
$$= \left\{E\left[\xi(X)^2\right] - E[\xi(X)]^2\right\}\left\{E\left[\eta(Y)^2\right] - E[\eta(Y)]^2\right\}$$
$$+E[\xi(X)]^2E\left[\eta(Y)^2\right] + E\left[\xi(X)^2\right]E[\eta(Y)]^2 - 2E[\xi(X)]^2E[\eta(Y)]^2 \tag{50}$$

Finally, we obtain,

$$\mathrm{Var}(\xi(X)\eta(Y)) = \mathrm{Var}(\xi(X))\mathrm{Var}(\eta(Y)) + E[\xi(X)]^2\mathrm{Var}(\eta(Y)) + \mathrm{Var}(\xi(X))E[\eta(Y)]^2 \tag{51}$$

or, by using matrix notations, we have

$$\mathrm{Var}(\xi(X)\eta(Y)) = \begin{pmatrix}\mathrm{Var}(\xi(X)) & E[\xi(X)]^2\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\mathrm{Var}(\eta(Y))\\ E[\eta(Y)]^2\end{pmatrix} \tag{52}$$

Especially

1. If $\xi(X) = \eta(Y) = 1$, then
   $E[\xi(X)] = E[\eta(Y)] = E[1] = 1.$ $\mathrm{Var}(\xi(X)) = \mathrm{Var}(\eta(Y)) = \mathrm{Var}(1) = 0,$
   By substituting the above relations into (51), we have
   $$\mathrm{Var}(1 \times 1) = \mathrm{Var}(1) = 0 \times 0 + 1 \times 0 + 0 \times 1 = 0 \tag{53}$$
2. If $\xi(X) = X,\ \eta(Y) = 1$, then
   $$E[\xi(X)] = E[X], \qquad \mathrm{Var}(\xi(X)) = \mathrm{Var}(X) \tag{54}$$
   $$E[\eta(Y)] = E[1] = 1, \mathrm{Var}(\eta(Y)) = \mathrm{Var}(1) = 0 \tag{55}$$
   and we obtain
   $$\mathrm{Var}(X \times 1) = \mathrm{Var}(X) \times 0 + E[X]^2 \times 0 + \mathrm{Var}(X) \times 1 \tag{56}$$
   $$= \mathrm{Var}(X). \tag{57}$$
3. If $\xi(X) = X,\ \eta(Y) = Y$, then
   $$E[\xi(X)] = E[X], \mathrm{Var}(\xi(X)) = \mathrm{Var}(X) \tag{58}$$
   $$E[\eta(Y)] = E[Y], \mathrm{Var}(\eta(Y)) = \mathrm{Var}(Y) \tag{59}$$
   and we obtain
   $$\mathrm{Var}(XY) = \mathrm{Var}(X)\mathrm{Var}(Y) + E[X]^2\mathrm{Var}(Y) + \mathrm{Var}(X)E[Y]^2. \tag{60}$$
4. If $\xi(X) = X,\ \eta(Y) = 1/Y$, then
   $$E[\xi(X)] = E[X], \qquad \mathrm{Var}(\xi(X)) = \mathrm{Var}(X) \tag{61}$$
   $$E[\eta(Y)] = E[1/Y], \quad \mathrm{Var}(\eta(Y)) = \mathrm{Var}(1/Y) \tag{62}$$
   and we have
   $$\mathrm{Var}(X/Y) = \mathrm{Var}(X)\mathrm{Var}(1/Y) + E[X]^2\mathrm{Var}(1/Y) + \mathrm{Var}(X)E[1/Y]^2. \tag{63}$$