Hidetoshi Takahashi "Electromagnetism"

高橋秀俊『電磁気学』(1959 裳華房,https://www.amazon.co.jp/dp/4785323035)
This text is excerpted and translated from pp.161-162 and pp.183-189.

III Magnetostatics

§ 68. Magnetic field and magnetic flux density

In the case of magnetism as well as in the case of electricity, we can think of the field of magnetic force, which is called the **magnetic field**, and customarily express its strength by H.

As Coulomb's law on magnetism shows that H is made with a central force, we see that H has a potential V_m

$$H = -\mathrm{grad}\, V_m \ . \qquad \text{(III-22)}$$

V_m is called the magnetic potential. V_m is obtained by multiplying the reciprocal of the distance from each magnetic pole by the strength of the magnetic pole and dividing by $4\pi\mu_0$

$$V_m = \frac{1}{4\pi\mu_0}\sum_i \frac{q_{mi}}{r_i} \ . \qquad \text{(III-23)}$$

H at a certain place is the force which works on the unit magnetic pole when it is placed in that place.However, sometimes there is a risk of changing the magnetic field by magnetizing the surrounding material by bringing the magnetic poles. In such a case, bring a minute magnetic pole that does not change the magnetic field, define

$$F = Hq_m \qquad \text{(III-24)}$$

as the force F acting on it, and define H as the limit value at which $q_m \to 0$ as the value of $H = F/\ q_m$.

Similarly, the work W necessary to move the magnetic pole q_m from the magnetic potential 0 to the magnetic potential V_m is given by

$$W = q_m V_m \ . \qquad \text{(III-24')}$$

IV Relationship between electricity and magnetism

Chapter 1 Electric Current and Magnetic Field

§ 79. Magnetic field created by electric current

The action of a magnet is formally similar in many ways to electricity. That is, it is a formal analogy in comparing static electricity and magnetostatics. On the other hand, the content relation between the two has not yet been established within the range described up to now. The study of magnetism was also relatively naive in the range where it was stuck only to the magnetostatics derived from the magnet. The importance of magnetism is expanding due to the direct relevance to electricity. Various analogies are effective due to various similarities between magnetism and electricity. By further stepping on, it has been long since anticipated that there is a direct action between the two. Franklin and Davy also mention this, but it was Oersted (1820) that described this in a clear form. He puts a magnetic needle **under a wire** facing the north and south, and when electric current is passed through the wire, the magnetic needle is veered. He expressed the phenomena by the law that **the magnetic pole on the cathode side** of the electric current always veers to the **west**. Next, if **the magnetic needle is placed on the wire**, the magnetic pole on the cathode side veers to the **east**. Moreover, he experimented with changing the direction of the wire variously, and found that the veer of the magnetic needle is proportional to the cosine (cos) of the angle between the direction of the wire and the north-south direction.

This magnetic action of electric current provides one new way to measure the electric current. Schweigger and Poggendorf made the galvanometer on this principle. Currently used galvanometers and ammeters are slightly different from this, but there is no difference in using magnetic action of electric current. Electric units currently in practical use (such as practical units) are all based on the magnetic action of electric current and are called **electromagnetic unit system**.

In terms of acting on the magnet, the electric current resembles the magnet itself. So the magnet gives force to another magnet, so it seems good that the magnet also gives force to the electric current, which is a kind of reaction. Further, the force between the electric currents is also conceivable.

§ 80. Ampère’s law

Thinking like this, Ampère conducted experiments. He did various experiments in a genius way and established the fundamental law of action between electric current and magnet and between electric currents of each other. This is an important law comparable to Coulomb's law.

Three facts are fundamental to that law.

① **The electric currents of two equal strengths running in mutually opposite directions cancel each other and do not produce a magnetic action to the outside.** That is, as shown in Fig. IV.1-1, if the wire is folded into two, no magnetic field appears outside.

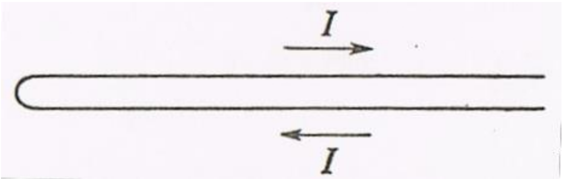

Fig. IV-1- 1 Magnetic fields of opposite electric currents of equal strengths cancel each other

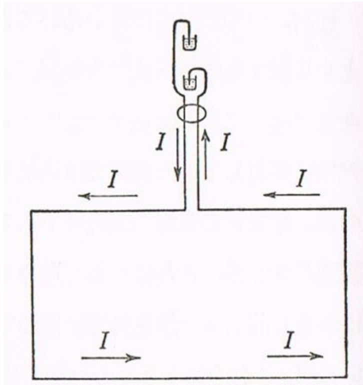

Fig. IV-1- 2 Face of the frame becomes perpendicular to the geomagnetism

② As shown in Fig. IV.1-2, the electric current is passed through the free-running wire frame, and the influence of geomagnetism on it is examined, so that the face of the frame is always oriented perpendicular to the north-south direction. Also, if the direction of the electric current is reversed, it will point in the opposite direction. That is, **the annular line through which the electric current flows behaves as if it were a magnet.**

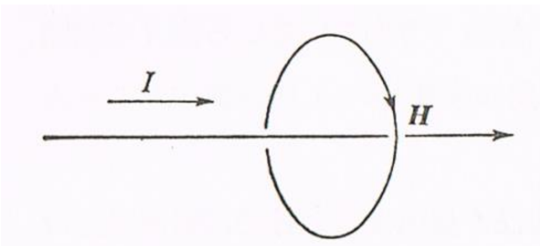

Fig. IV-1- 3 Electric current creates a circular magnetic field according to the right-hand screw law

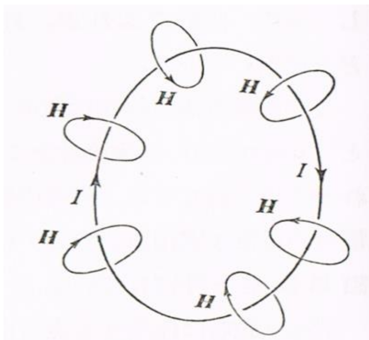

Fig. IV-1- 4 A magnetic field appears so that the S pole appears on the paper surface

③ In the experiment ②, the direction of the frame behaves as if **the direction of the screw is the N pole of the magnet when turning the right screw in the electric current direction** at any time. This may also be said that concentric magnetic fields are generated so that **the turning direction coincides with the direction of the magnetic field when turning the right hand screw so as to proceeding in the direction of electric current *I*** in Fig. IV.1-3. In the experiment ②, when electric current is passed through, the electric current in each part of the conductor produces a magnetic field and behaves as if it becomes a magnet. The direction of NS is obvious from Fig. IV.1-4.That is, in the case shown in the figure, S pole appears on the front side of the paper surface.

Therefore, Ampère thought as follows. According to ②, the force exerted by the outer magnet on the small toroidal electric current is equal to the force exerted on the small bar magnet facing the direction perpendicular to this ring. As a reaction of this force, this electric current should exert a force on the outer magnet. The force must be equal to the force that the small bar magnets assumed here exerts on the external magnets. In other words, this electric current ring must create a magnetic field similar to a small bar magnet.

Next, considering the larger annular electric current, this can be regarded as superimposition of small ring electric currents as shown in Fig. IV.1-5 according to the above property of ①. That is, the face surrounded by this large circle is regarded as a group of sections in which the electric current I circulates in a certain direction around each section obtained by dividing the original face into small pieces in the vertical and horizontal directions. Then, on the line at the boundary of the sections, the electric currents flowing around the sections on both sides of the sections just cancel each other, leaving only the influence of the electric current at the entire edge portion. In other words, the effect of this collection of small ring electric currents is the same as the action of the electric current I flowing through the entire edge portion.

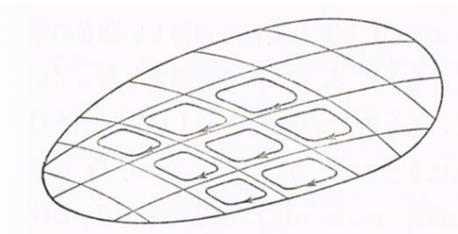

Fig. IV-1- 5

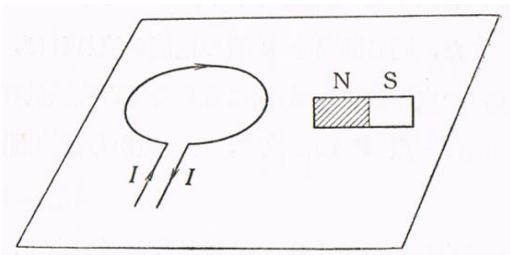

Fig. IV-1- 6 Magnetic potential in the plane including the coil is 0

Here, the effect of each small ring electric current is the same as placing a small bar magnet perpendicular to the face in that part, respectively. Therefore, the effect of the electric current flowing in the periphery on the outside has the same effect as that obtained by distributing magnets uniformly on an arbitrary curved surface surrounded by this electric current. That is, in other words, when there is an appropriate magnetic plate, a magnetic field is created such that the front and back surfaces are N and S poles, respectively. The strength of the magnetic moment per unit area of this **plate magnet** is constant anywhere on this plate. In particular, when the electric current flows along a curve on the plane, it is the same as a flat plate magnet with this curve as the periphery, and the magnetic moment of this magnet is proportional to the product of the electric current and the area of the curve.

Next, let us consider the distribution of the magnetic potential created by such an annular electric current. We explain this with the following idea (introduced by Prof. Tatsuoki Miyajima).

① When the electric current flows along the curve on the plane, the magnetic potential is 0 on this plane. The reason is as follows. At this time, all the dipoles of the magnet equivalent to the electric current are on this plane and are oriented perpendicularly to this plane so that all the magnetic fields they make are perpendicular to the plane on this plane. Therefore, we can see that the potential of the magnetic field created by the coil, i.e. the magnetic potential, is constant everywhere in that plane. However, considering that this plane is spreading infinitely far, the magnetic potential is 0 there, so the magnetic potential must be 0 everywhere on this plane.

② Next, **the magnetic potential that a certain coil makes at point P is proportional to the solid angle when looking at the coil from point P.**

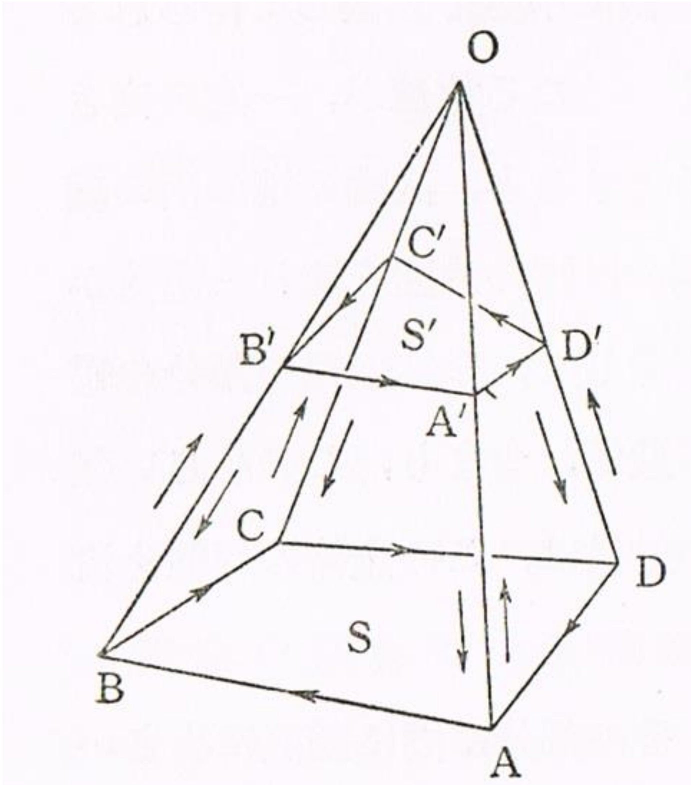

Fig. Ⅳ-1- 7 Magnetic potential at the point O due to the electric current S and S ' is 0

To prove them, let us consider the magnetic potential due to the electric currents flowing through the two coils S and S' shown in Figures IV.1-7. Here, if you see S, S' from the point O, if electric currents of equal strengths in opposite directions flow to each other as shown in the figure, and S and S' overlap at all(therefore, of course, the solid angles are equal each other), the magnetic potential at the point O is 0. The reason is as follows. Furthermore, we add two opposite electric currents flowing on the generatrix AA', BB', CC', DD' of pyramid with O at the apex (since these opposite electric currents cancel each other to 0, it is safe to assume that it exists). At that time, instead of the two electric currents flowing in S and S', we can think of four annular electric currents flowing in the directions of ABB'A', BCC'B', CDD'C', DAA'D' respectively. However, as mentioned above, since the point O is present on the plane containing these electric currents, the magnetic potential at the point O due to the respective currents is 0.

In this case, the above situation is true regardless of the shape of the electric current flowing circuit. So let's consider one circuit projected onto the surface of the unit sphere centered on point O as S'. Let us consider the electric current on the sphere as a small ring electric current similar to Fig. IV.1-5. If the ring electric current of any part on the sphere has the same area, the same contribution to the magnetic potential at the center. So in the end, it turns out that the magnetic potential of the point O is **proportional to the solid angle of the circuit as seen from the point O** in whatever form. However, when the solid angle including the circuit is defined as the area to cut off the unit sphere, when the solid angle including the circuit goes round the periphery of the surface on the spherical surface in the same direction as the electric current flowing through the circuit, it is decided to take the algebra sum of the area as positive if it is clockwise and negative if it turns counterclockwise.

In addition, it was experimentally confirmed that if the same circuit flows k times as much electric current, the magnetic force that it produces also becomes k times, so that the magnetic potential of any point also becomes k times. Therefore, from the above, when the solid angle of the electric current circuit viewed from that point is Ω, the magnetic potential of point P in the space can be written as

$$V_m = CI\Omega \ . \qquad \text{(IV.1-1)}$$

C is a proportional constant. The value of C changes depending on the way of unit of I and V_m. In general, it is widely practiced that this C is a number without dimension, thereby deriving the unit of the electric current I from the unit of the magnetic quantity, and such a unit system is called an **electromagnetic unit system**. The MKS rationalized unit system used in this book is also of this type, and it is stated here that C = 1 / 4π. Therefore,

$$V_m = \frac{1}{4\pi} I\Omega \, . \qquad \text{(IV.1-2)}$$

Since the unit of V_m is determined to be $\mu_0 = 4\pi \times 10^{-7}$ by the magnetic Coulomb's law, the unit of I will be determined from this, which is called **ampere**. Therefore, electric units, amperes, coulombs, etc. are essentially magnetically determined. This is probably due to the fact that almost all of the application and generation methods of electricity are related to electromagnetic phenomena, and the fact that electromagnetic methods are mainly used for measurement related to electricity. In fact, if we use electrostatic units, the everyday values of the amount of current etc. are inconvenient, bringing very large numbers.

As is evident from (IV.1-2) (because Ω has no dimension), the unit of the magnetic potential V_m is also ampere.

Thus, when considering the solid angle seen from O of a certain curve, it must be noted that there is arbitrariness. By projecting this curve onto the spherical surface, the spherical surface can be divided into two parts (in simple cases), but different values are obtained depending on which area of which is considered. If the area of one is S, the other is $4\pi R^2$ - S,

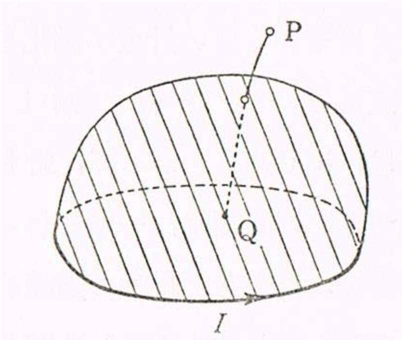

Fig. IV-1- 8 Add 4 π to the solid angle according to the number of times it passes through the curved surface bordering the loop

but since the direction in which currents flow around them is opposite, the latter should rather be written as $S' = S - 4\pi R^2$. Therefore the solid angle is

$$\Omega' = \frac{S'}{R^2} = \frac{S}{R^2} - 4\pi = \Omega - 4\pi, \qquad \text{(IV.1-3)}$$

which is different by 4π.

Now, when considering the magnetic potential, let us consider the magnetic potential of a magnet as being distributed on a certain edge which is the edge of the electric current circuit instead of the electric current. Therefore, from the point O, the solid angle on the side of this magnet surface should be taken. Only in that case the correct magnetic potential can be obtained. Therefore, when the point O passes through a certain face of this magnet, at that moment, the solid angle moves to the opposite side, so Ω will increase or decrease discontinuously by ± 4π.So then,

$$\frac{1}{4\pi} I' \left(\Omega' - \Omega \right) = \frac{1}{4\pi} I \times 4\pi = I. \qquad \text{(IV.1-4)}$$

In other words, **there is a discontinuity of only the magnetic potential *I* in the back side and the front side of the face of this magnet.**

By the way, in fact, it was mere fictitious for the sake of convenience that we thought that the magnets were distributed on the surface with the electric current circuit as the edge, and in reality, such a special surface does not exist anywhere . Therefore, it is not realistic that the magnetic potential becomes discontinuous in such a plane. However, if we do not think about such discontinuity, starting from the point P, once around the electric current flowing line and reaching the original point P, the magnetic potential increases or decreases by *I*. If it comes *n* times, the magnetic potential changes by *nI*. In other words, the magnetic potential at the same point in the space does not become one constant value, and it becomes a **multivalent function**. This is the same situation as Riemann surface in complex function theory. When there are several circuits, it turns them over many times to bring more complicated multivalency.

In other words, the magnetic field produced by the magnet was a place of preservation with a clear potential, but **the magnetic field produced by the electric current has no spatial monovalent potential**. This means that even if the magnetic pole q_m starts from one point of the magnetic field and returns to its original position all around, the sum of work

$$\oint q_m H \, ds \qquad \text{(IV.1-5)}$$

does not become 0. According to the law of conservation of energy, this energy must be supplied from somewhere. Actually, this energy is obtained from a power supply that supplies electric current to the circuit by electromagnetic induction to be described later.

On the other hand, it is necessary to note that the magnetic poles are never actually present alone and necessarily accompany N and S poles. The magnetic potential at a certain point in the space is not measured alone, and only the difference between the magnetic potential of a certain place of the N pole and the magnetic potential of the S pole is always a problem. So, for example, suppose that a soft wire is magnetized, and N and S poles are present at both ends. The S pole is fixed to one point. Let's move the N pole first at point P, then turn around the electric current and return to point P. At this time, the wire is stuck in the electric current circuit once, differently from the beginning. In other words, even if the magnetic pole of "sticking" comes around once to the original position, it never returns to the same state as the original. Therefore, as long as there is no single magnetic pole, inconvenience never occurs even if the magnetic potential is not a monovalent function.

Writing the above result into the equation is

$$\oint H \cdot ds = nI \ . \qquad \text{(IV.1-6)}$$

Here, n is the number of times the integrating path is wound a circuit through which the electric current passes.

The number n of how many times the integration path C and the current circuit S are entangled is a so-called **topological quantity**. If both of these curves are continuously deformed (without cutting or intersecting), the number n does not change. If $n \neq 0$, these two rings are entangled and can never leave. If one side (for example, C) is stretched out, S sticks in the opposite way. In that case, S wraps C n times, that is, from the topological point of view, there is no distinction which is wound, which is a mutual equal relationship.

When there are many conducting wires through which the electric current flows, the magnetic potential resulting therefrom is the sum of the magnetic potentials produced by the respective electric currents, so that the magnetic field H is also the sum of the respective magnetic fields $H_1 \ldots H_n$. Considering the integral

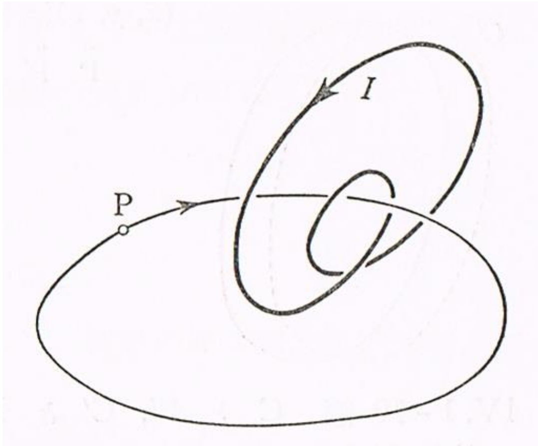

Fig. IV-1- 9 Passing through the twice winding coil once is equivalent to twisting the winding coil twice

$$\oint H\, ds = \sum_i \oint H_i ds \ , \qquad \text{(IV.1-7)}$$

that goes around a certain closed curve C, if the ith electric current is I_i and this current is entangled with the curve C n_i times (if the direction of current is reversed, n_i is negative)

$$\oint H\, ds = \sum_i n_i I_i \ . \qquad \text{(IV.1-8)}$$

Of course, $n_i = 0$ for the electric current not entangled with C. That is, the right side of (IV.1-8) means the algebraic sum of the electric currents flowing through the curved surface with the curve C as an edge.

(IV.1-8) determines the amount of electric current flowing through any curved surface in space when given the field of H. Therefore, contrary to (IV.1-2), it can be regarded as an equation for obtaining the electric current from the magnetic field. In this sense, (IV.1-8) is an important equation playing a role corresponding to the Gauss' theorem (I.3-25) in static electricity in the magnetic field theory.